HOCHSCHILD PRODUCTS AND GLOBAL NON-ABELIAN COHOMOLOGY FOR ALGEBRAS. APPLICATIONS

A. L. AGORE AND G. MILITARU

Abstract. Let A be a unital associative algebra over a field k, E a vector space and $\pi : E \to A$ a surjective linear map with $V = \mathrm{Ker}(\pi)$. All algebra structures on E such that $\pi : E \to A$ becomes an algebra map are described and classified by an explicitly constructed global cohomological type object $\mathbb{GH}^2\,(A,\,V)$. Any such algebra is isomorphic to a Hochschild product $A \star V$, an algebra introduced as a generalization of a classical construction. We prove that $\mathbb{GH}^2\,(A,\,V)$ is the coproduct of all non-abelian cohomologies $\mathbb{H}^2\,(A,\,(V,\cdot))$. The key object $\mathbb{GH}^2\,(A,\,k)$ responsible for the classification of all co-flag algebras is computed. All Hochschild products $A \star k$ are also classified and the automorphism groups $\mathrm{Aut}_{\mathrm{Alg}}(A \star k)$ are fully determined as subgroups of a semidirect product $A^* \ltimes \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big)$ of groups. Several examples are given as well as applications to the theory of supersolvable coalgebras or Poisson algebras. In particular, for a given Poisson algebra P, all Poisson algebras having a Poisson algebra surjection on P with a 1-dimensional kernel are described and classified.

Introduction

Introduced at the level of groups by Hölder [28], the extension problem is a famous and still open problem to which a vast literature was devoted (see [1] and the references therein). Fundamental results obtained for groups [1, 15, 37] served as a model for studying the extension problem for several other fields such as Lie/Leibniz algebras [14, 31], super Lie algebras [6], associative algebras [17, 26], Hopf algebras [8], Poisson algebras [24], Lie-Rinehart algebras [12, 25] etc. The extension problem is one of the main tools for classifying 'finite objects' and has been a source of inspiration for developing cohomology theories in all fields mentioned above. We recall the extension problem using the language of category theory. Let $\mathcal{C}$ be a category having a zero (i.e. an initial and final) object 0 and for which it is possible to define an exact sequence. Given A, B two *fixed* objects of $\mathcal{C}$, the extension problem consists of the following question:

Describe and classify all *extensions of A by B*, i.e. all triples $(E,\,i,\,\pi)$ consisting of an object E of $\mathcal{C}$ and two morphisms in $\mathcal{C}$ that fit into an exact sequence of the form:

$$0 \longrightarrow B \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } A \longrightarrow 0$$

2010 *Mathematics Subject Classification.* 16D70, 16S70, 16E40.

Key words and phrases. The extension problem, non-abelian cohomology, classification of algebras.

A.L. Agore is Postdoctoral Fellow of the Fund for Scientific Research-Flanders (Belgium) (F.W.O. Vlaanderen). This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, grant no. 88/05.10.2011.

Two extensions (E, i, π) and (E', i', π') of A by B are called *equivalent* if there exists an isomorphism $\varphi : E \to E'$ in $\mathcal{C}$ that stabilizes B and co-stabilizes A, i.e. $\varphi \circ i = i'$ and $\pi' \circ \varphi = \pi$. The answer to the extension problem is given by explicitly computing the set $\mathrm{Ext}(A, B)$ of all equivalence classes of extensions of A by B via this equivalence relation. The simplest case is that of extensions with an 'abelian' kernel B for which a Schreier type theorem proves that all extensions of A by an abelian object B are classified by the second cohomology group $\mathrm{H}^2(A, B)$ - the result is valid for groups, Lie/Leibniz/associative/Poisson/Hopf algebras, but the construction of the second cohomology group is different for each of the above categories [8, 14, 15, 26, 37]. The difficult part of the extension problem is the case when B is not abelian: as a general principle, the Schreier type theorems remain valid, but this time the classifying object of all extensions of A by B is not the cohomology group of a given complex anymore, but only a pointed set called the *non-abelian cohomology* $\mathrm{H}^2_{\mathrm{nab}}(A, B)$. For its construction in the case of groups we refer to [7], while for Lie algebras to [19] where it was proved that the non-abelian cohomology $\mathrm{H}^2_{\mathrm{nab}}(A, B)$ is the Deligne groupoid of a suitable differential graded Lie algebra. The difficulty of the problem consists in explicitly computing $\mathrm{H}^2_{\mathrm{nab}}(A, B)$: lacking an efficient cohomology tool as in the abelian case [35, 38], it needs to be computed 'case by case' using different computational and combinatorial approaches.

This paper deals, at the level of associative algebras, with a generalization of the extension problem, called the *global extension (GE) problem*, introduced recently in [3, 33] for Poisson/Leibniz algebras as a categorical dual of the extending structures problem [2, 4, 5]. The GE-problem can be formulated for any category $\mathcal{C}$ using a simple idea: in the classical extension problem we drop the hypothesis 'B is a fixed object in $\mathcal{C}$' and replace it by a weaker one, namely 'B has a fixed dimension'. For example, if $\mathcal{C}$ is the category of unital associative algebras over a field k, the GE-problem can be formulated as follows: for a given algebra A, classify all associative algebras E for which there exists a surjective algebra map $E \to A \to 0$ whose kernel has a given dimension $\mathfrak{c}$ as a vector space. Of course, any such algebra has $A \times V$ as the underlying vector space, where V is a vector space such that $\dim(V) = \mathfrak{c}$. Among several equivalent possibilities for writing down the GE-problem for algebras, we prefer the following:

Let A be a unital associative algebra, E a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces. Describe and classify the set of all unital associative algebra structures that can be defined on E such that $\pi : E \to A$ becomes a morphism of algebras.

By classification of two algebra structures $\cdot_E$ and $\cdot'_E$ on E we mean the classification up to an isomorphism of algebras $(E, \cdot_E) \cong (E, \cdot'_E)$ that stabilizes $V := \mathrm{Ker}(\pi)$ and co-stabilizes A: we shall denote by $\mathrm{Gext}(A, E)$ the set of equivalence classes of all algebra structures on E such that $\pi : E \to A$ is an algebra map. Let us explain now the significant differences between the GE-problem and the classical extension problem for associative algebras whose study was initiated in [17, 26]. Let $(E, \cdot_E)$ be a unital algebra structure on E such that $\pi : (E, \cdot_E) \to A$ is an algebra map. Then $(E, \cdot_E)$ is an extension of the unital algebra A by the associative algebra $V = \mathrm{Ker}(\pi)$, which is a non-unital subalgebra (in fact a two-sided ideal) of $(E, \cdot_E)$. However, the multiplication on V is not fixed from the input data, as in the case of the classical extension problem: it depends essentially on the algebra structures on E which we are looking for. Thus the classical extension problem is in some sense the *'local'* version of the GE-problem. The partial answer and the best result obtained so far for the classical extension problem was given in [26, Theorem 6.2]: all algebra structures $\cdot_E$ on E such that V is a two-sided ideal of null square (i.e. $x \cdot_E y = 0$, for all $x,\, y \in V$ - that is V is an *'abelian'* algebra) are classified by the second Hochschild cohomological group $\mathrm{H}^2(A,\, V)$. The result, as remarkable as it is, has its limitations: for example, if $A := \mathrm{M}_n(k)$ is the algebra of $n \times n$-matrices then there is no associative algebra E of dimension $1 + n^2$ for which we have a surjective algebra map $A \to \mathrm{M}_n(k)$ with a null square kernel - however, there is a large family of such algebras with a projection on $\mathrm{M}_n(k)$, one of them being of course the direct product of algebras $\mathrm{M}_n(k) \times k$ (more details are given in Example 2.13). This example provides enough motivation for studying the GE-problem and the fact that it covers the missing part in the classical Hochschild approach of the extension problem.

The paper is organized as follows. Section 1 gives the theoretical answer to the GE-problem in three steps. First of all, in Proposition 1.2 we introduce a new product $A \star V = A \star_{(\triangleleft,\, \triangleright,\, \vartheta,\, \cdot)} V$, associated to an algebra A and a vector space V connected by two 'actions' of A on V, a 'cocycle' and an associative multiplication $\cdot$ on V satisfying several axioms. We call the algebra $A \star V$ the *Hochschild product* since, in the particular case when $\cdot$ is the trivial multiplication on V the above product reduces to the one introduced by Hochschild in [26]: in this case the axioms involved in the construction of $A \star V$ come down to the fact that V is an A-bimodule and $\vartheta : A \times A \to V$ is an usual normalized 2-cocycle. On the other hand, if the cocycle ϑ is the trivial map then the associated Hochschild product $A \star V$ is just the *semidirect product* $A \# V$ of algebras and the corresponding axioms show that $(V,\, \cdot)$ is an associative algebra in the monoidal category $({}_A\mathcal{M}_A,\, - \otimes_A -,\, A)$ of A-bimodules. The canonical surjection $A \star V \to A$ is an algebra map having the kernel V. Proposition 1.4 proves the converse: any algebra structure $\cdot_E$ which can be defined on the vector space E such that $\pi : (E, \cdot_E) \to A$ is a morphism of algebras is isomorphic to a Hochschild product $A \star V$. Based on these results and a technical lemma, the theoretical answer to the GE-problem is given in Theorem 1.9: the classifying set $\mathrm{Gext}(A,\, E)$ is parameterized by an explicitly constructed global cohomological object $\mathbb{GH}^2\,(A,\, V)$ and the bijection between the elements of $\mathbb{GH}^2\,(A,\, V)$ and $\mathrm{Gext}(A,\, E)$ is given. On the route, Corollary 1.5 proves that any finite dimensional algebra is isomorphic to an iteration of Hochschild products of the form $\big(\cdots\big((S \star V_1) \star V_2\big) \star \cdots \star V_t\big)$, where S is a finite dimensional simple algebra and $V_1, \cdots, V_t$ are finite dimensional vector spaces. Corollary 1.11 provides a decomposition of $\mathbb{GH}^2\,(A,\, V)$ as the coproduct of all non-abelian cohomologies $\mathbb{H}^2\,\big(A,\, (V,\, \cdot_V)\big)$, which are classifying objects for the extensions of A by all associative algebra structures $\cdot$ on V – the second Hochschild cohomological group $\mathrm{H}^2\,(A,\, V)$ is the most elementary piece among all components of $\mathbb{GH}^2\,(A,\, V)$. Computing the classifying object $\mathbb{GH}^2\,(A,\, V)$ is a highly nontrivial task: if A is finite dimensional and $V := k^n$ this object parameterizes the equivalence classes of all unital associative algebras of dimension $n + \dim(A)$ that admit an algebra surjection on A. In Section 2 we shall identify a way of computing this object for a class of algebras called *co-flag* algebras over A, i.e. algebras E that have a finite chain of surjective morphisms of algebras $A_n := E \xrightarrow{\pi_n} A_{n-1} \cdots \xrightarrow{\pi_2} A_1 \xrightarrow{\pi_1} A_0 := A$, such that $\dim(\mathrm{Ker}(\pi_i)) = 1$, for all $i = 1, \cdots, n$. All co-flag algebras over A can be completely described and classified by a recursive reasoning whose key step is given in Proposition 2.6 and Corollary 2.7 where $\mathbb{GH}^2\,(A,\,k)$ is computed and all Hochschild products $A \star k$ are described by generators and relations. The less restrictive classification of these algebras, i.e. only up to an agebra isomorphism, is given in Theorem 2.8 where the second classifying object $\mathbb{HOC}\,(A,\,k)$ is computed: it parameterizes the isomorphism classes of all Hochschild products $A \star k$, that is, it classifies up to an isomorphism, all algebras B which admit a surjective algebra map $B \to A$ with a 1-dimensional kernel. As a bonus of our approach, the automorphism groups $\mathrm{Aut}_{\mathrm{Alg}}(A \star k)$ are fully determined in Corollary 2.12 as subgroups of a semidirect product $A^* \ltimes \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big)$ of groups. At this point we should mention that determining the structure of the automorphism group of a given algebra is an old problem, intensively studied and very difficult, arising from invariant theory (see [10, 13] and the references therein). Several examples where both classifying objects $\mathbb{GH}^2\,(A,\,k)$ and $\mathbb{HOC}\,(A,\,k)$ are explicitly computed for different algebras A are worked out in details. For instance, if $A = k[C_n]$, is the group algebra of the cyclic group C_n of order n, then $\mathbb{GH}^2\,(k[C_n],\,k) \cong \big(U_n(k) \times U_n(k)\big) \sqcup k^*$, where $U_n(k)$ is the group of n-th roots of unity in k. The classification of these algebras, by describing the classifying object $\mathbb{HOC}\,(k[C_n],\,k)$, is also indicated and reduces the question to a challenging number theory problem which depends heavily on the arithmetics of n and the base field k – it is also related to two intensively studied problems in the theory of group rings, namely the description of all invertible elements and the automorphism group of a group algebra [23, 32, 34]. An intriguing example is $A := \mathcal{T}_n(k)$, the algebra of upper triangular $(n \times n)$-matrices. The global cohomological object $\mathbb{GH}^2\,(\mathcal{T}_n(k),\,k)$ is computed in Example 2.19 being described by a very interesting set of matrices of trace 0. In particular, $|\mathbb{HOC}\,(\mathcal{T}_2(k),\,k)| = 8$, i.e. up to an isomorphism of algebras there exist exactly eight 4-dimensional algebras that have an algebra projection on the Heisenberg algebra $\mathcal{T}_2(k)$. Applications for coalgebras and Poisson algebras are given in Section 3 based on the same idea: namely, that of rephrasing the concepts and results of this paper for coalgebras (resp. Poisson manifolds) via two different contravariant functors $(-)^* := \mathrm{Hom}_k\,(-,\,k)$ (resp. $C^\infty(-)$) from the category of coalgebras (resp. Poisson manifolds) to the category of algebras (resp. Poisson algebras). Having the theory of supersolvable Lie algebras [9] as a source of inspiration we introduce the concept of a *supersolvable coalgebra* in such a way that a coalgebra C is supersolvable if and only if the convolution algebra C^* is a co-flag algebra. In particular, Corollary 2.18 classifies all 3-dimensional supersolvable coalgebras over a field of characteristic $\neq 2$. On the other hand, for a given Poisson algebra P, Theorem 3.3 classifies up to an isomorphism all Poisson algebras Q which admit a Poisson surjection $Q \to P \to 0$ with a 1-dimensional kernel. The result is the algebraic counterpart of the classification problem of all Poisson manifolds containing a given Poisson manifold M of codimension 1. As an example, we show that there exist exactly six families of 4-dimensional Poisson algebras with a Poisson algebra surjection of the Heisenberg-Poisson algebra $\mathcal{H}(3,\,k)$. For applications and further motivation for studying Poisson algebras we refer to [21, 22, 30].

1. The global extension problem

Notations and terminology. For two sets X and Y we shall denote by $X \sqcup Y$ their coproduct in the category of sets, i.e. $X \sqcup Y$ is the disjoint union of X and Y. Unless otherwise specified, all vector spaces, linear or bilinear maps are over an arbitrary field k. A map $f : V \to W$ between two vector spaces is called the trivial map if $f(v) = 0$, for all $v \in V$. By an algebra A we always mean a unital associative algebra over k whose unit will be denoted by 1_A. All algebra maps preserve units and any left/right A-module is unital. For an algebra A, $\mathrm{Aut}_{\mathrm{Alg}}(A)$ denotes the group of algebra automorphisms of A, $\mathrm{Alg}\,(A,\, k)$ is the space of all algebra maps $A \to k$ while ${}_A\mathcal{M}_A$ stands for the category of A-bimodules. If $(V,\, \triangleright,\, \triangleleft) \in {}_A\mathcal{M}_A$ is an A-bimodule, then the *trivial extension* of A by V is the algebra $A \times V$, with the multiplication defined for any $a,\, b \in A$, $x,\, y \in V$ by:

$$(a,\, x) \cdot (b,\, y) := \big(ab,\;\, a \triangleright y + x \triangleleft b\big) \tag{1}$$

Let A be an algebra, E a vector space, $\pi : E \to A$ a linear epimorphism of vector spaces with $V := \mathrm{Ker}(\pi)$ and denote by $i : V \to E$ the inclusion map. We say that a linear map $\varphi : E \to E$ *stabilizes* V (resp. *co-stabilizes* A) if the left square (resp. the right square) of the following diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "Id"'] & E \arrow[r, "\pi"] \arrow[d, "\varphi"] & A \arrow[d, "Id"] \\ V \arrow[r, "i"] & E \arrow[r, "\pi"] & A \end{tikzcd} \tag{2}$$

is commutative. Two unital associative algebra structures $\cdot$ and $\cdot'$ on E such that $\pi : E \to A$ is a morphism of algebras are called *cohomologous* and we denote this by $(E, \cdot) \approx (E, \cdot')$, if there exists an algebra map $\varphi : (E, \cdot) \to (E, \cdot')$ which stabilizes V and co-stabilizes A. We can easily prove that any such morphism is bijective and thus, $\approx$ is an equivalence relation on the set of all algebra structures on E such that $\pi : E \to A$ is an algebra map and we denote by $\mathrm{Gext}\,(A,\, E)$ the set of all equivalence classes via the equivalence relation $\approx$. $\mathrm{Gext}\,(A,\, E)$ is the classifying object for the GE-problem. In what follows we will prove that $\mathrm{Gext}\,(A,\, E)$ is parameterized by a global cohomological object $\mathbb{GH}^2\,(A,\, V)$ which will be explicitly constructed. To start with, we introduce the following:

Definition 1.1. Let A be an algebra and V a vector space. A *Hochschild data* of A by V is a system $\Theta(A, V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ consisting of four bilinear maps

$$\triangleright\; : A \times V \to V, \quad \triangleleft : V \times A \to V, \quad \vartheta\; : A \times A \to V, \quad \cdot : V \times V \to V$$

For a Hochschild data $\Theta(A, V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ we denote by $A \star V = A \star_{(\triangleleft,\, \triangleright,\, \vartheta,\, \cdot)} V$ the vector space $A \times V$ with the multiplication defined for any $a,\, b \in A$ and $x,\, y \in V$ by:

$$(a, x) \star (b, y) := (ab,\; \vartheta(a, b) + a \triangleright y + x \triangleleft b + x \cdot y) \tag{3}$$

$A \star V$ is called the *Hochschild product* associated to $\Theta(A, V)$ if it is an associative algebra with the multiplication given by (3) and the unit $(1_A,\, 0_V)$. In this case $\Theta(A, V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ is called a *Hochschild system* of A by V. The multiplication defined by (3) is more general than the one appearing in the proof of [26, Theorem 6.2] – the latter arises as a special case of $A \star V$ for $\cdot : V \times V \to V$ the trivial map, that is $x \cdot y = 0$, for all $x,\, y \in V$. Moreover, the GE-problem is the dual, in the sense of category theory, of the extending structures problem studied for algebras in [5]: hence, from this viewpoint the Hochschild product $A \star V$ can be seen as a categorical dual of the *unified product* $A \ltimes V$ introduced in [5, Theorem 2.2]. The necessary and sufficient conditions for $A \star V$ to be a Hochschild product are given in the following:

Proposition 1.2. *Let A be an algebra, V a vector space and $\Theta(A, V) = (\triangleright, \triangleleft, \vartheta, \cdot)$ a Hochschild data of A by V. Then $A \star V$ is a Hochschild product if and only if the following compatibility conditions hold for any $a,\, b,\, c \in A$ and $x,\, y \in V$:*

- (H0) $\vartheta(a,\, 1_A) = \vartheta(1_A,\, a) = 0, \quad x \triangleleft 1_A = x, \quad 1_A \triangleright x = x$
- (H1) $(x \cdot y) \triangleleft a = x \cdot (y \triangleleft a)$
- (H2) $(x \triangleleft a) \cdot y = x \cdot (a \triangleright y)$
- (H3) $a \triangleright (x \cdot y) = (a \triangleright x) \cdot y$
- (H4) $(a \triangleright x) \triangleleft b = a \triangleright (x \triangleleft b)$
- (H5) $\vartheta(a,\, bc) - \vartheta(ab,\, c) = \vartheta(a,\, b) \triangleleft c - a \triangleright \vartheta(b,\, c)$
- (H6) $(ab) \triangleright x = a \triangleright (b \triangleright x) - \vartheta(a,\, b) \cdot x$
- (H7) $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b - x \cdot \vartheta(a,\, b)$
- (H8) *The bilinear map $\cdot : V \times V \to V$ is associative.*

Before going into the proof, we make some comments on the relations (H0)-(H8). The first relation in (H0) together with (H5) show that ϑ is a normalized Hochschild 2-cocycle. (H6) and (H7) are deformations of the usual left and respectively right A-module conditions: together with (H4) and the last two relations of (H0) they measure how far $(V,\, \triangleright,\, \triangleleft)$ is from being an A-bimodule. Finally, axioms (H1)-(H3) are compatibilities between the associative multiplication $\cdot$ on V and the 'actions' $(\triangleright,\, \triangleleft)$ of A on V which are missing in the classical theory [26] since $\cdot$ is the trivial map and thus they are automatically fulfilled. When ϑ is the trivial map, axioms (H1)-(H3) together with (H6)-(H8) imply that $(V, \cdot)$ is an associative algebra in the monoidal category ${}_A\mathcal{M}_A = ({}_A\mathcal{M}_A,\, - \otimes_A -,\, A)$ of A-bimodules (see Example 1.3 below).

Proof. To start with, we can easily prove that $(1_A,\, 0_V)$ is the unit for the multiplication defined by (3) if and only if (H0) holds. The rest of the proof relies on a detailed analysis of the associativity condition for the multiplication given by (3). Since in $A \star V$ we have $(a, x) = (a, 0) + (0, x)$, it follows that the associativity condition holds if and only if it holds for all generators of $A \star V$, i.e. for the set $\{(a,\, 0) \mid a \in A\} \cup \{(0,\, x) \mid x \in V\}$. To save space we will illustrate only a few cases, the rest of the details being left to the reader. For instance, the associativity condition for the multiplication given by (3) holds in {(0, x), (0, y), (a, 0)} if and only if (H1) holds. Similarly, the associativity condition holds in {(0, x), (a, 0), (0, y)} if and only if (H2) holds while, the associativity condition holds in {(0, x), (0, y), (0, z)} if and only if $\cdot : V \times V \to V$ is associative. □

From now on a Hochschild system of A by V will be viewed as a system of bilinear maps $\Theta(A, V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ satisfying the axioms (H0)-(H8) and we denote by $\mathcal{HS}\,(A,\, V)$ the set consisting of all Hochschild systems of A by V.

Examples 1.3. 1. By applying Proposition 1.2 we obtain that a Hochschild data $(\triangleright, \triangleleft, \vartheta, \cdot)$ for which $\cdot$ is the trivial map is a Hochschild system if and only if $(V, \triangleright, \triangleleft)$ is an A-bimodule and $\vartheta : A \times A \to V$ is a normalized 2-cocycle. Furthermore, if ϑ is also the trivial map, then the associated Hochschild product is just the trivial extension of the algebra A by the A-bimodule V as defined by (1).

2. A Hochschild system $\Theta(A, V) = (\triangleright, \triangleleft, \vartheta, \cdot)$ for which ϑ is the trivial map is called a *semidirect system* of A by V. In this case ϑ will be omitted when writing down $\Theta(A, V)$ and axioms in Proposition 1.2 take a simplified form: $\Theta(A, V) = (\triangleright, \triangleleft, \cdot)$ is a semidirect system if and only if $(V, \triangleright, \triangleleft) \in {}_A\mathcal{M}_A$ is an A-bimodule, $(V, \cdot)$ is an associative algebra and

$$(x \cdot y) \triangleleft a = x \cdot (y \triangleleft a), \qquad a \triangleright (x \cdot y) = (a \triangleright x) \cdot y, \qquad (x \triangleleft a) \cdot y = x \cdot (a \triangleright y) \tag{4}$$

for all $a \in A$, x, $y \in V$. The Hochschild product associated to a semidirect system $\Theta(A, V) = (\triangleright, \triangleleft, \cdot)$ is called a *semidirect product* of algebras and will be denoted by $A \# V := A \#_{(\triangleleft, \triangleright, \cdot)} V$. The terminology will be motivated below in Corollary 1.6: exactly as in the case of groups or Lie algebras, the semidirect product of algebras describes split epimorphisms in the category of algebras. We will rephrase the axioms of a semidirect system $\Theta(A, V) = (\triangleright, \triangleleft, \cdot)$ of A by V using the language of monoidal categories. The first and the second axioms of (4) are equivalent to the fact that the bilinear map $\cdot : V \times V \to V$ is an A-bimodule map, while the last one is the same as saying that the map is A-balanced. The space of these maps is in one-to-one correspondence with the set of all A-bimodule maps $V \otimes_A V \to V$. This fact together with the other two axioms can be rephrased as follows: $\Theta(A, V) = (\triangleright, \triangleleft, \cdot)$ is a semidirect system of A by V if and only if $(V, \cdot)$ is a (not-necessarily unital) associative algebra in the monoidal category ${}_A\mathcal{M}_A = ({}_A\mathcal{M}_A, - \otimes_A -, A)$ of A-bimodules.

The Hochschild product is the tool to answer the GE-problem. Indeed, first we observe that the canonical projection $\pi_A : A \star V \to A$, $\pi_A(a, x) := a$ is a surjective algebra map with kernel $\{0\} \times V \cong V$. Hence, the algebra $A \star V$ is an extension of the algebra A by the associative algebra $(V, \cdot)$ via

$$0 \longrightarrow V \xrightarrow{i_V} A \star V \xrightarrow{\pi_A} A \longrightarrow 0 \tag{5}$$

where $i_V(x) = (0, x)$. Conversely, we have:

Proposition 1.4. *Let A be an algebra, E a vector space and $\pi : E \to A$ an epimorphism of vector spaces with $V = \mathrm{Ker}(\pi)$. Then any algebra structure $\cdot$ which can be defined on the vector space E such that $\pi : (E, \cdot) \to A$ becomes a morphism of algebras is isomorphic to a Hochschild product $A \star V$ and moreover, the isomorphism of algebras $(E, \cdot) \cong A \star V$ can be chosen such that it stabilizes V and co-stabilizes A.*

Thus, any unital associative algebra structure on E such that $\pi : E \to A$ is an algebra map is cohomologous to an extension of the form (5).

Proof. Let $\cdot$ be an algebra structure of E such that $\pi : (E, \cdot) \to A$ is an algebra map. Since k is a field we can pick a k-linear section $s : A \to E$ of π, i.e. $\pi \circ s = \mathrm{Id}_A$ and $s(1_A) = 1_E$. Then $\varphi : A \times V \to E$, $\varphi(a, x) := s(a) + x$ is an isomorphism of vector spaces with the inverse $\varphi^{-1}(y) = \big(\pi(y),\, y - s(\pi(y))\big)$, for all $y \in E$. Using the section s we define three bilinear maps given for any $a,\, b \in A$ and $x \in V$ by:

$$\begin{aligned} \triangleleft &= \triangleleft_s \; : V \times A \to V, \quad x \triangleleft a := x \cdot s(a) \\ \triangleright &= \triangleright_s \; : A \times V \to V, \quad a \triangleright x := s(a) \cdot x \\ \vartheta &= \vartheta_s \; : A \times A \to V, \quad \vartheta(a, b) := s(a) \cdot s(b) - s(ab) \end{aligned}$$

and let $\cdot_V : V \times V \to V$ be the restriction of $\cdot$ at V, i.e. $x \cdot_V y := x \cdot y$, for all $x,\, y \in V$. We can easily see that they are well-defined maps. The key step is the following: using the system $(\triangleleft,\, \triangleright,\, \vartheta,\, \cdot_V = \cdot)$ connecting A and V we can prove that the unique algebra structure $\star$ that can be defined on the direct product of vector spaces $A \times V$ such that $\varphi : A \times V \to (E, \cdot)$ is an isomorphism of algebras is given by:

$$(a, x) \star (b, y) := (ab,\, \vartheta(a, b) + a \triangleright y + x \triangleleft b + x \cdot y) \tag{6}$$

for all $a,\, b \in A,\, x,\, y \in V$. Indeed, let $\star$ be such an algebra structure on $A \times V$. Then:

$$\begin{aligned} (a, x) \star (b, y) &= \varphi^{-1}\big(\varphi(a, x) \cdot \varphi(b, y)\big) = \varphi^{-1}\big((s(a) + x) \cdot (s(b) + y)\big) \\ &= \varphi^{-1}(s(a) \cdot s(b) + s(a) \cdot y + x \cdot s(b) + x \cdot y) \\ &= \big(ab,\, s(a) \cdot s(b) - s(ab) + s(a) \cdot y + x \cdot s(b) + x \cdot y\big) \\ &= \big(ab,\, \vartheta(a, b) + a \triangleright y + x \triangleleft b + x \cdot y\big) \end{aligned}$$

as needed. Thus, $\varphi : A \star V \to (E, \cdot)$ is an isomorphism of algebras and we can see that it stabilizes V and co-stabilizes A. □

Using Proposition 1.4 we obtain the following result concerning the structure of finite dimensional algebras which indicates the crucial role played by Hochschild products. We can survey all algebras of a given dimension if we are able to compute various Hochschild systems starting with a simple algebra (whose structure is well-known due to the Wederburn-Artin theorem) and the associated Hochschild products. It is the associative algebra counterpart of a similar result from group theory [36, pages 283-284].

Corollary 1.5. *Any finite dimensional algebra is isomorphic to an iteration of Hochschild products of the form* $\Big(\cdots\big((S \star V_1) \star V_2\big) \star \cdots \star V_t\Big)$, *where* S *is a finite dimensional simple* k*-algebra,* t *is a positive integer and* $V_1, \cdots, V_t$ *are finite dimensional vector spaces.*

Proof. Let A be an algebra of dimension n. The proof goes by induction on n. If $n = 1$ then $A \cong k \cong k \star \{0\}$ and k is a simple algebra. Assume now that $n > 1$. If A is simple there is nothing to prove. On the contrary, if A has a proper two-sided ideal $\{0\} \neq V_t \neq A$, let $\pi : A \to A_1 := A/V_t$ be the canonical projection. It follows from Proposition 1.4 that $A \cong A_1 \star V_t$, for some Hochschild system of A_1 by V_t. If A_1 is simple the proof is finished; if A_1 is not simple, we apply induction since $\dim_k(A_1) < n$. □

The semidirect products of algebras characterize split epimorphism in this category:

Corollary 1.6. *An algebra map* $\pi : B \to A$ *is a split epimorphism in the category of algebras if and only if there exists an isomorphism of algebras* $B \cong A\#V$, *where* $V = \mathrm{Ker}(\pi)$ *and* $A\#V$ *is a semidirect product of algebras.*

Proof. First we note that for a semidirect product $A\#V$, the canonical projection $p_A : A\#V \to A$, $p_A(a,\, x) = a$ has a section that is an algebra map defined by $s_A(a) = (a, 0)$, for all $a \in A$. Conversely, let $s : A \to B$ be an algebra map such that $\pi \circ s = \mathrm{Id}_A$. Then, the bilinear map ϑ_s constructed in the proof of Proposition 1.4 is the trivial map and hence the corresponding Hochschild product $A \star V$ is a semidirect product $A\#V$. □

Proposition 1.4 shows that the classification part of the GE-problem reduces to the classification of all Hochschild products associated to all Hochschild systems of A by V. This is what we will do next.

Lemma 1.7. *Let $\Theta(A,V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ and $\Theta'(A,V) = (\triangleright',\, \triangleleft',\, \vartheta',\, \cdot')$ be two Hochschild systems and $A{\star}V$, respectively $A{\star'}V$, the corresponding Hochschild products. Then there exists a bijection between the set of all morphisms of algebras $\psi : A \star V \to A \star' V$ which stabilize V and co-stabilize A and the set of all linear maps $r : A \to V$ with $r(1_A) = 0$ satisfying the following compatibilities for all $a,\, b \in A,\, x,\, y \in V$:*

(CH1) $x \cdot y = x \cdot' y$;
(CH2) $x \triangleleft a = x \triangleleft' a + x \cdot' r(a)$;
(CH3) $a \triangleright x = a \triangleright' x + r(a) \cdot' x$;
(CH4) $\vartheta(a,\, b) + r(ab) = \vartheta'(a,\, b) + a \triangleright' r(b) + r(a) \triangleleft' b + r(a) \cdot' r(b)$

Under the above bijection the morphism of algebras $\psi = \psi_r : A{\star}V \to A{\star'}V$ corresponding to $r : A \to V$ is given by $\psi(a,x) = (a,\, r(a) + x)$, for all $a \in A$, $x \in V$. Moreover, $\psi = \psi_r$ is an isomorphism with the inverse given by $\psi_r^{-1} = \psi_{-r}$.

Proof. It is an elementary fact that a linear map $\psi : A \times V \to A \times V$ stabilizes V and co-stabilizes A if and only if there exists a uniquely determined linear map $r : A \to V$ such that $\psi(a,x) = (a, r(a) + x)$, for all $a \in A$, $x \in V$. Let $\psi = \psi_r$ be such a linear map. We will prove that $\psi : A \star V \to A \star' V$ is an algebra map if and only if $r(1_A) = 0$ and the compatibility conditions (CH1)-(CH4) hold. To start with it is straightforward to see that ψ preserve the unit $(1_A,\, 0)$ if and only if $r(1_A) = 0$. The proof will be finished if we check that the following compatibility holds for all generators of $A \times V$:

$$\psi\big((a,x) \star (b,y)\big) = \psi\big((a,x)\big) \star' \psi\big((b,y)\big) \tag{7}$$

By a straightforward computation it follows that (7) holds for the pair $(a,0)$, $(b,0)$ if and only if (CH4) is fulfilled while (7) holds for the pair $(0,x)$, $(a,0)$ if and only if (CH2) is satisfied. Finally, (7) holds for the pair $(a,0)$, $(0,x)$ and respectively $(0,x)$, $(0,y)$ if and only if (CH3) and respectively (CH1) hold. □

Lemma 1.7 leads to the following:

Definition 1.8. Let A be an algebra and V a vector space. Two Hochschild systems $\Theta(A,V) = (\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ and $\Theta'(A,V) = (\triangleright',\, \triangleleft',\, \vartheta',\, \cdot')$ are called *cohomologous*, and we denote this by $\Theta(A,V) \approx \Theta'(A,V)$, if and only if $\cdot = \cdot'$ and there exists a linear map

$r : A \to V$ such that $r(1_A) = 0$ and for any $a, b \in A$, $x, y \in V$ we have:

$$x \triangleleft a = x \triangleleft' a + x \cdot' r(a) \tag{8}$$

$$a \triangleright x = a \triangleright' x + r(a) \cdot' x \tag{9}$$

$$\vartheta(a, b) = \vartheta'(a, b) - r(ab) + a \triangleright' r(b) + r(a) \triangleleft' b + r(a) \cdot' r(b) \tag{10}$$

As a conclusion of this section, we obtain the theoretical answer to the GE-problem:

Theorem 1.9. *Let A be an algebra, E a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces with $V = \mathrm{Ker}(\pi)$. Then $\approx$ defined in Definition 1.8 is an equivalence relation on the set $\mathcal{HS}(A, V)$ of all Hochschild systems of A by V. If we denote by $\mathbb{GH}^2(A, V) := \mathcal{HS}(A, V)/\approx$, then the map*

$$\mathbb{GH}^2(A, V) \to \mathrm{Gext}(A, E), \qquad \overline{(\triangleright, \triangleleft, \vartheta, \cdot)} \longmapsto A \star_{(\triangleleft, \triangleright, \vartheta, \cdot)} V \tag{11}$$

is bijective, where $\overline{(\triangleright, \triangleleft, \vartheta, \cdot)}$ denotes the equivalence class of $(\triangleright, \triangleleft, \vartheta, \cdot)$ via $\approx$.

Proof. Follows from Proposition 1.2, Proposition 1.4 and Lemma 1.7. □

Computing the classifying object $\mathbb{GH}^2(A, V)$, for a given algebra A and a given vector space V is a very difficult task. The first step in decomposing this object is suggested by the way the equivalence relation $\approx$ was introduced in Definition 1.8: it shows that two different associative algebra structures $\cdot$ and $\cdot'$ on V give two different equivalence classes of the relation $\approx$ on $\mathcal{HS}(A, V)$. Let us fix $\cdot_V$ an associative multiplication on V and denote by $\mathcal{HS}_{\cdot_V}(A, V)$ the set of all triples $\big(\triangleleft, \triangleright, \vartheta\big)$ such that $\big(\triangleright, \triangleleft, \vartheta, \cdot_V\big) \in \mathcal{HS}(A, V)$. Two triples $\big(\triangleleft, \triangleright, \vartheta\big)$ and $\big(\triangleleft', \triangleright', \vartheta'\big) \in \mathcal{HS}_{\cdot_V}(A, V)$ are $\cdot_V$-cohomologous and we denote this by $\big(\triangleleft, \triangleright, \vartheta\big) \approx_{\cdot_V} \big(\triangleleft', \triangleright', \vartheta'\big)$ if there exists a linear map $r : A \to V$ such that $r(1_A) = 0$ and the compatibility conditions (8)-(10) are fulfilled for $\cdot' = \cdot_V$. Then $\approx_{\cdot_V}$ is an equivalence relation on $\mathcal{HS}_{\cdot_V}(A, V)$ and we denote by $\mathbb{H}^2\big(A, (V, \cdot_V)\big)$ the quotient set $\mathcal{HS}_{\cdot_V}(A, V)/\approx_{\cdot_V}$. The non-abelian cohomology $\mathbb{H}^2\big(A, (V, \cdot_V)\big)$ classifies all extensions of the unital associative algebra A by a fixed associative algebra $(V, \cdot_V)$. This can be seen as a Schreier type theorem for associative algebras: we mention that [26, Theorem 6.2] (see also [35, Proposition 3.7]) follows as a special case of Corollary 1.10 if we let $\cdot_V$ to be the trivial map.

Corollary 1.10. *Let A be a unital associative algebra and $(V, \cdot_V)$ an associative multiplication on V. Then, the map*

$$\mathbb{H}^2\big(A, (V, \cdot_V)\big) \to \mathrm{Ext}(A, (V, \cdot_V)), \qquad \overline{\overline{(\triangleright, \triangleleft, \vartheta)}} \longmapsto A \star_{(\triangleleft, \triangleright, \vartheta, \cdot_V)} V \tag{12}$$

is bijective, where $\mathrm{Ext}(A, (V, \cdot_V))$ is the set of equivalence classes of all unital associative algebras that are extensions of the algebra A by $(V, \cdot_V)$ and $\overline{\overline{(\triangleright, \triangleleft, \vartheta)}}$ denotes the equivalence class of $(\triangleright, \triangleleft, \vartheta)$ via $\approx_l$.

The above considerations give also the following decomposition of $\mathbb{GH}^2(A, V)$:

Corollary 1.11. *Let A be an algebra, E a vector space and $\pi : E \to A$ an epimorphism of vector spaces with $V = \mathrm{Ker}(\pi)$. Then*

$$\mathbb{GH}^2(A, V) = \sqcup_{\cdot_V} \mathbb{H}^2\big(A, (V, \cdot_V)\big) \tag{13}$$

where the coproduct on the right hand side is in the category of sets over all possible associative algebra structures $\cdot_V$ *on the vector space* V*.*

By looking at formula (13) one can see that computing $\mathbb{GH}^2(A,\,V)$ is a very laborious task in which the first major barrier is describing all associative multiplications on V. The complexity of the computations involved increases along side with $\dim(V)$. Among all components of the coproduct in (13) the simplest one is that corresponding to the trivial associative algebra structure on V, i.e. $x \cdot_V y := 0$, for all $x,\, y \in V$. We shall denote this trivial algebra structure on V by $V_0 := (V,\, \cdot_V = 0)$ and we shall prove that $\mathbb{H}^2\big(A,\, V_0\big)$ is the coproduct of all classical second cohomological groups. Indeed, let $\mathcal{HS}_0(A,\, V_0)$ be the set of all triples $\big(\triangleleft,\, \triangleright,\, \vartheta\big)$ such that $\big(\triangleright,\, \triangleleft,\, \vartheta,\, \cdot_V := 0\big) \in \mathcal{HS}(A, V)$. Example 1.3 shows that a triple $\big(\triangleleft,\, \triangleright,\, \vartheta\big) \in \mathcal{HS}_0(A,\, V_0)$ if and only if $(V, \triangleright, \triangleleft)$ is an A-bimodule and $\vartheta : A \times A \to V$ is a normalized 2-cocycle. Two triples $\big(\triangleleft,\, \triangleright,\, \vartheta\big)$ and $\big(\triangleleft',\, \triangleright',\, \vartheta'\big) \in \mathcal{HS}_0(A,\, V_0)$ are 0-cohomologous $\big(\triangleleft,\, \triangleright,\, \vartheta\big) \approx_0 \big(\triangleleft',\, \triangleright',\, \vartheta'\big)$ if and only if $\triangleleft = \triangleleft'$, $\triangleright = \triangleright'$ and there exists a linear map $r : A \to V$ such that $r(1_A) = 0$ and

$$\vartheta(a,\, b) = \vartheta'(a,\, b) - r(ab) + a \triangleright r(b) + r(a) \triangleleft b \tag{14}$$

for all $a,\, b \in A$ – these are the conditions remaining from Definition 1.8 applied for the trivial multiplication $\cdot := 0$. The equalities $\triangleleft = \triangleleft'$ and $\triangleright = \triangleright'$ show that two different A-bimodule structures over V give different equivalence classes in the classifying object $\mathbb{H}^2\big(A,\, V_0\big)$. Thus, for computing it we can also fix $(V,\, \triangleleft,\, \triangleright)$ an A-bimodule structure over V and consider the set $\mathrm{Z}^2_{(\triangleleft,\, \triangleright)}(A,\, V_0)$ of all normalized Hochschild 2-cocycles: i.e. bilinear maps $\vartheta : A \times A \to V$ satisfying (H5) and the first condition of (H0). Two normalized 2-cocycles ϑ and ϑ' are cohomologous $\vartheta \approx_0 \vartheta'$ if and only if there exists a linear map $r : A \to V$ such that $r(1_A) = 0$ and (14) holds. $\approx_0$ is an equivalence relation on the set $\mathrm{Z}^2_{(\triangleleft,\, \triangleright)}(A,\, V_0)$ and the quotient set $\mathrm{Z}^2_{(\triangleleft,\, \triangleright)}(A,\, V_0)/\approx_0$ is just the classical second Hochschild cohomological group which we denote by $\mathrm{H}^2_{(\triangleleft,\, \triangleright)}(A,\, V_0)$. All the above considerations prove the following:

Corollary 1.12. *Let* A *be an algebra and* V *a vector space viewed with the trivial associative algebra structure* V_0*. Then:*

$$\mathbb{H}^2\big(A,\, V_0\big) = \sqcup_{(\triangleleft,\, \triangleright)} \mathrm{H}^2_{(\triangleleft,\, \triangleright)}(A,\, V_0) \tag{15}$$

where the coproduct on the right hand side is in the category of sets over all possible A*-bimodule structures* $(\triangleleft,\, \triangleright)$ *on the vector space* V*.*

2. Co-flag algebras. Examples.

In this section we apply the theoretical results obtained in Section 1 for some concrete examples: more precisely, for a given algebra A we shall classify all unital associative algebras B such that there exists a surjective algebra map $\pi : B \to A$ having a 1-dimensional kernel, which as a vector space will be assumed to be k. First, we shall compute $\mathbb{GH}^2(A,\, k)$: it will classify all these algebras up to an isomorphism which stabilizes k and co-stabilizes A. Then, we will compute the second classifying object, denoted by $\mathbb{HOC}(A,\, k)$, which will provide the classification of these algebras only up to an isomorphism. Computing both classifying objects is the key step in a recursive algorithm for describing and classifying the new class of algebras defined as follows:

Definition 2.1. Let A be an algebra and E a vector space. A unital associative algebra structure $\cdot_E$ on E is called a *co-flag algebra over* A if there exists a positive integer n and a finite chain of surjective morphisms of algebras

$$A_n := (E, \cdot_E) \xrightarrow{\pi_n} A_{n-1} \xrightarrow{\pi_{n-1}} A_{n-2} \cdots \xrightarrow{\pi_2} A_1 \xrightarrow{\pi_1} A_0 := A \tag{16}$$

such that $\dim_k(\mathrm{Ker}(\pi_i)) = 1$, for all $i = 1, \cdots, n$. A finite dimensional algebra is called a *co-flag algebra* if it is a co-flag algebra over the unital algebra k.

By applying successively Proposition 1.4 we obtain that a co-flag algebra over an algebra A is isomorphic to an iteration of Hochschild products of the form $\big(\cdots\big((A\star k)\star k\big)\star\cdots\star k\big)$, where the 1-dimensional vector space k appears n times in the above product. The tools used for describing co-flag algebras are the following:

Definition 2.2. Let A be an algebra. A *co-flag datum of the first kind of* A is a triple $(\lambda, \Lambda, \vartheta)$ consisting of two algebra maps[1] λ, $\Lambda : A \to k$ and a bilinear map $\vartheta : A \times A \to k$ satisfying the following compatibilities for any a, b, $c \in A$:

$$\vartheta(a, 1_A) = \vartheta(1_A, a) = 0, \qquad \vartheta(a, bc) - \vartheta(ab, c) = \vartheta(a, b)\Lambda(c) - \vartheta(b, c)\lambda(a) \tag{17}$$

A *co-flag datum of the second kind of* A is a pair (λ, u) consisting of a linear map $\lambda : A \to k$ such that $\lambda(1_A) = 1$ and a non-zero scalar $u \in k^*$.

We denote by $\mathcal{CF}_1(A)$ (resp. $\mathcal{CF}_2(A)$) the set of all co-flag data of the first (resp. second) kind of A and by $\mathcal{CF}(A) := \mathcal{CF}_1(A) \sqcup \mathcal{CF}_2(A)$ their coproduct; the elements of $\mathcal{CF}(A)$ are called *co-flag data* of A. The set of co-flag data $\mathcal{CF}(A)$ parameterizes the set of all Hochschild systems of A by a 1-dimensional vector space. The next result also describes the first algebra A_1 from the exact sequence (16) in terms depending only on A.

Proposition 2.3. *Let A be an algebra. Then there exists a bijection $\mathcal{HS}(A, k) \cong \mathcal{CF}(A)$ between the set of all Hochschild systems of A by k and the set of all co-flag data of A given such that the Hochschild product $A \star k$ associated to $(\lambda, \Lambda, \vartheta) \in \mathcal{CF}_1(A)$ is the algebra denoted by $A_{(\lambda,\Lambda,\vartheta)}$ with the multiplication given for any a, $b \in A$, x, $y \in k$ by:*

$$(a, x) \star (b, y) = \big(ab,\ \vartheta(a, b) + \lambda(a)y + \Lambda(b)x\big) \tag{18}$$

while the Hochschild product $A \star k$ associated to $(\lambda, u) \in \mathcal{CF}_2(A)$ is the algebra denoted by $A^{(\lambda,u)}$ with the multiplication given for any a, $b \in A$, x, $y \in k$ by:

$$(a, x) \star (b, y) = \big(ab,\ u^{-1}\big(\lambda(a)\lambda(b) - \lambda(ab)\big) + \lambda(a)y + \lambda(b)x + u\,xy\big) \tag{19}$$

Proof. We have to compute the set of all bilinear maps $\triangleright : A \times k \to k$, $\triangleleft : k \times A \to k$, $\vartheta : A \times A \to k$ and $\cdot : k \times k \to k$ satisfying the compatibility conditions (H0)-(H8) of Proposition 1.2. Since k has dimension 1 there exists a bijection between the set of all Hochschild datums $\big(\triangleright, \triangleleft, \vartheta, \cdot\big)$ of A by k and the set of all 4-tuples $(\Lambda, \lambda, \vartheta, u)$ consisting of two linear maps Λ, $\lambda : A \to k$, a bilinear map $\vartheta : A \times A \to k$ and a scalar

[1] Recall that we assume the algebra maps $\lambda : A \to k$ to be unit preserving, i.e. $\lambda(1_A) = 1$.

$u \in k$. The bijection is given such that the Hochschild datum $(\triangleright,\, \triangleleft,\, \vartheta,\, \cdot)$ corresponding to $(\Lambda,\, \lambda,\, \vartheta,\, u)$ is defined as follows:

$$a \triangleright x := \lambda(a)x, \quad x \triangleleft a := \Lambda(a)x, \quad x \cdot y := u\, xy$$

for all $a \in A$ and $x,\, y \in k$. Now, axiom (H0) holds if and only if $\vartheta(a, 1_A) = \vartheta(1_A, a) = 0$ and $\lambda(1_A) = \Lambda(1_A) = 1$. Axioms (H1), (H3), (H4) and (H8) are trivially fulfilled. Axiom (H5) is equivalent to $\vartheta(a, bc) - \vartheta(ab, c) = \vartheta(a, b)\Lambda(c) - \vartheta(b, c)\lambda(a)$, axiom (H6) is equivalent to

$$\lambda(ab) = \lambda(a)\lambda(b) - u\,\vartheta(a, b) \tag{20}$$

while axiom (H7) is equivalent to $\Lambda(ab) = \Lambda(a)\Lambda(b) - u\,\vartheta(a, b)$. Finally, axiom (H2) is equivalent to $u\Lambda(a) = u\lambda(a)$, for all $a \in A$. A discussion on u is imposed by the last compatibility condition and the conclusion follows easily: $\mathcal{CF}_1\,(A)$ corresponds to the case when $u = 0$ and this will give rise to the algebras $A \star k = A_{(\lambda, \Lambda, \theta)}$. The case $\mathcal{CF}_2\,(A)$ corresponds to $u \neq 0$; in this case $\Lambda = \lambda$ and the cocycle ϑ is implemented by u and λ via the formula $\vartheta(a, b) := u^{-1}\big(\lambda(a)\lambda(b) - \lambda(ab)\big)$, for all $a,\, b \in A$, that arises from (20). Moreover, we can easily check that axiom (17) is trivially fulfilled for ϑ defined as above. The algebra $A^{(\lambda, u)}$ is just the Hochschild product $A \star k$ associated to this context. □

Remarks 2.4. (1) The first family of Hochschild products $A_{(\lambda, \Lambda, \theta)}$ constructed in Proposition 2.3 corresponds to the classical case in which $k \cong 0 \times k$ is a two-sided ideal of null square in the algebra $A_{(\lambda, \Lambda, \theta)}$. The algebras $A_{(\lambda, \Lambda, \theta)}$ will be classified up to an isomorphism in Theorem 2.8 below. For the new families of algebras $A^{(\lambda, u)}$ the kernel of the canonical projection $\pi_A : A^{(\lambda, u)} \to k$ is equal to $k \cong 0 \times k$ and this is not a null square ideal since $(0, 1) \star (0, 1) = (0,\, u) \neq (0, 0)$. Let $(\lambda, u) \in \mathcal{CF}_2\,(A)$ be a co-flag datum of the second kind of A. Taking into account the multiplication on $A^{(\lambda,\, u)}$ given by (19) we can easily prove that the map:

$$\varphi : A^{(\lambda,\, u)} \to A \times k, \qquad \varphi(a, x) := (a,\, \lambda(a) + u\, x) \tag{21}$$

for all $a \in A$ and $x \in k$ is an isomorphism of algebras (which does not stabilize k, if $u \neq 1$), where $A \times k$ is the usual direct product of algebras. The inverse of φ is given by $\varphi^{-1}(a, x) = \big(a,\, u^{-1}(x - \lambda(a))\big)$, for all $a \in A$ and $x \in k$.

We will now describe the algebra $A_{(\lambda, \Lambda, \vartheta)}$ and $A^{(\lambda, u)}$ by generators and relations. The elements of A will be seen as elements in $A \times k$ via the identification $a = (a,\, 0)$ and we denote by $f := (0_A,\, 1) \in A \times k$. Let $\{e_i \,|\, i \in I\}$ be a basis of A as a vector space over k. Then the algebra $A_{(\lambda, \Lambda, \vartheta)}$ is the vector space having $\{f,\, e_i \,|\, i \in I\}$ as a basis and the multiplication $\star$ given for any $i \in I$ by:

$$e_i \star e_j = e_i \cdot_A e_j + \vartheta(e_i,\, e_j)\, f, \;\; f^2 = 0, \;\; e_i \star f = \lambda(e_i)\, f, \;\; f \star e_i = \Lambda(e_i)\; f \tag{22}$$

where $\cdot_A$ denotes the multiplication on A. The algebra $A^{(\lambda, u)}$ is the vector space having $\{f,\, e_i \,|\, i \in I\}$ as a basis and the multiplication $\star$ given for any $i \in I$ by:

$$e_i \star e_j = e_i \cdot_A e_j + u^{-1}\big(\lambda(e_i)\lambda(e_j) - \lambda(e_i \cdot_A e_j)\big)\, f, \;\; f^2 = u\, f, \;\; e_i \star f = f \star e_i = \lambda(e_i)\, f \tag{23}$$

Using Proposition 2.3, Proposition 1.4 and the isomorphism $A^{(\lambda,\, u)} \simeq A \times k$ we obtain:

Corollary 2.5. *Let A be an algebra. A unital associative algebra B has a surjective algebra map $B \to A \to 0$ whose kernel is 1-dimensional if and only if B is isomorphic to $A \times k$ or $A_{(\lambda,\Lambda,\vartheta)}$, for some $(\lambda, \Lambda, \vartheta) \in \mathcal{CF}_1\,(A)$.*

We are now able to compute the classifying object $\mathbb{GH}^2\,(A,\, k)$.

Proposition 2.6. *Let A be an algebra. Then,*

$$\mathbb{GH}^2\,(A,\, k) \cong \Big(\mathcal{CF}_1\,(A)/\approx_1\Big) \sqcup k^*$$

where $\approx_1$ is the following equivalence relation on $\mathcal{CF}_1\,(A)$: $(\lambda, \Lambda, \vartheta) \approx_1 (\lambda', \Lambda', \vartheta')$ if and only if $\lambda = \lambda'$, $\Lambda = \Lambda'$ and there exists a linear map $t : A \to k$ such that for any $a,\, b \in A$:

$$\vartheta(a, b) = \vartheta'(a, b) - t(ab) + \lambda'(a)t(b) + \Lambda'(b)t(a) \tag{24}$$

Proof. It follows from Theorem 1.9 and Proposition 2.3 that

$$\mathbb{GH}^2\,(A,\, k) \cong \Big(\mathcal{CF}_1\,(A)/\approx_1\Big) \sqcup \Big(\mathcal{CF}_2\,(A)/\approx_2\Big)$$

where the equivalence relation $\approx_i$ on $\mathcal{CF}_i\,(A)$, for $i = 1, 2$, is just the equivalence relation $\approx$ from Definition 1.8 written for the sets $\mathcal{CF}_i\,(A)$ via the bijection $\mathcal{HS}\,(A,\, k) \cong \mathcal{CF}\,(A)$ given in Proposition 2.3. The equivalence relation $\approx$ written on the set of all co-flag data of the first kind takes precisely the form from the statement – we mention that a linear map t satisfying (24) has the property that $t(1_A) = 0$. The equivalence relation $\approx$ written on $\mathcal{CF}_2\,(A)$ takes the following form: $(\lambda, u) \approx_2 (\lambda', u')$ if and only if $u = u'$ and there exists a linear map $t : A \to k$ such that for any $a \in A$ we have:

$$\lambda(a) = \lambda'(a) + t(a)\, u' \tag{25}$$

Now, if we fix a unit preserving linear map $\lambda^0 : A \to k$ we obtain that the set $\{(\lambda^0,\, u) \,|\, u \in k^*\}$ is a system of representatives for the equivalence relation $\approx_2$ on $\mathcal{CF}_2\,(A)$ and hence $\mathcal{CF}_2\,(A)/\approx_2\, \cong\, k^*$, which finishes the proof. □

The way $\approx_1$ is defined in Proposition 2.6 indicates the decomposition of $\Big(\mathcal{CF}_1\,(A)/\approx_1\Big)$ as follows: for two fixed algebra maps $(\lambda,\, \Lambda) \in \mathrm{Alg}(A,\, k)$ we shall denote by $\mathrm{Z}^2_{(\lambda,\,\Lambda)}\,(A,\, k)$ the set of all normalized $(\lambda,\, \Lambda)$-cocycles; that is, the set of all bilinear maps $\vartheta : A \times A \to k$ satisfying the following compatibilities for any $a,\, b,\, c \in A$:

$$\vartheta(a, 1_A) = \vartheta(1_A, a) = 0, \qquad \vartheta(a, bc) - \vartheta(ab, c) = \vartheta(a, b)\Lambda(c) - \vartheta(b, c)\lambda(a)$$

Two $(\lambda,\, \Lambda)$-cocycles $\vartheta,\, \vartheta' : A \times A \to k$ are equivalent $\vartheta \approx_1^{(\lambda,\,\Lambda)} \vartheta'$ if and only if there exists a linear map $t : A \to k$ such that

$$\vartheta(a, b) = \vartheta'(a, b) - t(ab) + \lambda(a)t(b) + \Lambda(b)t(a) \tag{26}$$

for all $a,\, b \in A$. If we denote $\mathrm{H}^2_{(\lambda,\,\Lambda)}\,(A,\, k) := \mathrm{Z}^2_{(\lambda,\,\Lambda)}\,(A,\, k)/\approx_1^{(\lambda,\,\Lambda)}$ we obtain the following decomposition of $\mathbb{GH}^2\,(A,\, k)$:

Corollary 2.7. *Let A be an algebra. Then,*

$$\mathbb{GH}^2\,(A,\,k)\;\cong\;\Big(\sqcup_{(\lambda,\Lambda)}\;\mathrm{H}^2_{(\lambda,\,\Lambda)}\,(A,\,k)\Big)\sqcup k^* \tag{27}$$

where the coproduct on the right hand side is in the category of sets over all possible algebra maps $\lambda,\,\Lambda : A \to k$.

The classifying object $\mathbb{GH}^2\,(A,\,k)$ computed in Corollary 2.7 classifies all Hochschild products $A \star k$ up to an isomorphism of algebras which stabilizes k and co-stabilizes A. In what follows we will consider a less restrictive classification: we denote by $\mathbb{HOC}\,(A,\,k)$ the set of algebra isomorphism classes of all Hochschild products $A{\star}k$. Two cohomologous Hochschild products $A \star k$ and $A \star' k$ are of course isomorphic and therefore there exists a canonical projection $\mathbb{GH}^2\,(A,\,k) \twoheadrightarrow \mathbb{HOC}\,(A,\,k)$ between the two classifying objects. Next we compute $\mathbb{HOC}\,(A,\,k)$.

Theorem 2.8. *Let A be an algebra. Then there exists a bijection:*

$$\mathbb{HOC}\,(A,\,k) \cong \Big(\mathcal{CF}_1\,(A)/\equiv\Big)\sqcup\{\,A\times k\,\} \tag{28}$$

where $\equiv$ is the equivalence relation on $\mathcal{CF}_1\,(A)$ defined by: $(\lambda,\Lambda,\vartheta)\equiv(\lambda',\Lambda',\vartheta')$ if and only if there exists a triple $(s_0,\,\psi,\,r)\in k^*\times \mathrm{Aut}_{\mathrm{Alg}}(A)\times \mathrm{Hom}_k(A,\,k)$ consisting of a non-zero scalar $s_0\in k^*$, an algebra automorphism ψ of A and a linear map $r : A \to k$ such that for any $a,\,b\in A$ we have:

$$\lambda = \lambda'\circ\psi,\qquad \Lambda=\Lambda'\circ\psi \tag{29}$$

$$\vartheta(a,b)\,s_0 = \vartheta'\big(\psi(a),\psi(b)\big){+}\lambda(a)r(b)+\Lambda(b)r(a)-r(ab) \tag{30}$$

Proof. Corollary 2.5 shows that any Hochschild product $A{\star}k$ is isomorphic to $A_{(\lambda,\Lambda,\vartheta)}$, for some $(\lambda,\Lambda,\vartheta)\in\mathcal{CF}_1\,(A)$ or to $A^{(\lambda',u')}$, for some $(\lambda',u')\in\mathcal{CF}_2\,(A)$. Since $A^{(\lambda',u')}\cong A\times k$, the proof relies on the following two steps:

(1) Let $(\lambda,\Lambda,\vartheta)$ and $(\lambda',\Lambda',\vartheta')\in\mathcal{CF}_1\,(A)$. Then, there exists a bijection between the set of all algebra isomorphisms $\varphi : A_{(\lambda,\Lambda,\vartheta)}\to A_{(\lambda',\Lambda',\vartheta')}$ and the set of all triples $(s_0,\,\psi,\,r)\in k^*\times\mathrm{Aut}_{\mathrm{Alg}}(A)\times\mathrm{Hom}_k(A,\,k)$ satisfying the compatibility conditions (29) and (30). The bijection is given such that the algebra isomorphism $\varphi=\varphi_{(s_0,\,\psi,\,r)}$ associated to $(s_0,\,\psi,\,r)$ is defined for any $a\in A$ and $x\in k$ by:

$$\varphi_{(s_0,\,\psi,\,r)}(a,\,x) = \big(\psi(a),\,r(a)+xs_0\big) \tag{31}$$

(2) The algebras $A_{(\lambda,\Lambda,\vartheta)}$ and $A^{(\lambda',u')}\cong A\times k$ are not isomorphic.

We start by proving (1); although this is more than we need for proving our theorem, this more general statement will be used later on in computing the automorphism groups for the algebras $A_{(\lambda,\Lambda,\vartheta)}$. First we note that there exists a bijection between the set of all linear maps $\varphi : A\times k\to A\times k$ and the set of quadruples $(s_0,\,\beta_0,\,\psi,\,r)\in k\times A\times \mathrm{Hom}_k(A,\,A)\times\mathrm{Hom}_k(A,\,k)$ given such that the linear map $\varphi=\varphi_{(s_0,\,\beta_0,\,\psi,\,r)}$ associated to $(s_0,\,\beta_0,\,\psi,\,r)$ is given for any $a\in A$ and $x\in k$ by:

$$\varphi(a,\,x) = \big(\psi(a)+x\,\beta_0,\,r(a)+x\,s_0\big) \tag{32}$$

We will prove now the following technical fact: a linear map given by (32) is an isomorphism of algebras from $A_{(\lambda,\Lambda,\vartheta)}$ to $A_{(\lambda',\Lambda',\vartheta')}$ if and only if $\beta_0 = 0$, $s_0 \neq 0$, ψ is an algebra automorphism of A and (29)-(30) hold. Taking into account the multiplication on $A_{(\lambda,\Lambda,\vartheta)}$ given by (18), we can easily obtain that $\varphi\big((0,x) \star (0,y)\big)= \varphi(0,x) \star' \varphi(0,x)$ if and only if $\beta_0 = 0$, where by $\star'$ we denote the multiplication of $A_{(\lambda',\Lambda',\vartheta')}$. Hence, in order for φ to be an algebra map it should take the following simplified form for any $a \in A$ and $x \in k$:

$$\varphi(a,\, x) = \big(\psi(a),\, r(a) + x\, s_0\big) \tag{33}$$

for some triple $(s_0,\, \psi,\, r) \in k \times \mathrm{Hom}_k(A,\, A) \times \mathrm{Hom}_k(A,\, k)$. Next we prove that a linear map given by (33) is an algebra morphism from $A_{(\lambda,\Lambda,\vartheta)}$ to $A_{(\lambda',\Lambda',\vartheta')}$ if and only if $\psi : A \to A$ is an algebra map and the following compatibilities are fulfilled for any a, $b \in A$:

$$\lambda(a)\, s_0 = \lambda'\big(\psi(a)\big)\, s_0, \qquad \Lambda(a)\, s_0 = \Lambda'\big(\psi(a)\big)\, s_0 \tag{34}$$

$$r(ab) + \vartheta(a,b)\, s_0 = \vartheta'\big(\psi(a), \psi(b)\big) + \lambda'(\psi(a))r(b) + \Lambda'(\psi(b))r(a) \tag{35}$$

Indeed, φ preserves the unit $(1_A, 0)$ if and only if $\psi(1_A) = 1_A$ and $r(1_A) = 0$. On the other hand we can prove that the first (resp. the second) compatibility of (34) is exactly the condition $\varphi\big((a,0) \star (0,x)\big)= \varphi(a,0) \star' \varphi(0,x)$ (resp. $\varphi\big((0,x) \star (a,0)\big)= \varphi(0,x) \star' \varphi(a,0)$). Finally, the condition $\varphi\big((a,0) \star (b,0)\big)= \varphi(a,0) \star' \varphi(b,0)$ is equivalent to the fact that ψ is an algebra endomorphism of A and (35) holds. Finally, the condition $r(1_A) = 0$ follows by taking $a = b = 1_A$ in (35). Step (1) is finished if we prove that an algebra map $\varphi = \varphi_{(s_0,\, \psi,\, r)}$ given by (33) is bijective if and only if $s_0 \neq 0$ and ψ is an automorphism of A. Assume first that $s_0 \neq 0$ and ψ is bijective with the inverse ψ^{-1}. Then, we can see that $\varphi_{(s_0,\, \psi,\, r)}$ is an isomorphism of algebras with the inverse given by $\varphi^{-1}_{(s_0,\, \psi,\, r)} := \varphi_{(s_0^{-1},\, \psi^{-1},\, -(r\circ\psi^{-1})s_0^{-1})}$. Conversely, assume that φ is bijective. Then its inverse φ^{-1} is an algebra map and thus has the form $\varphi^{-1}(a,\, x) = (\psi'(a),\, r'(a) + xs'_0)$, for some triple (s'_0, r', ψ'). If we write $\varphi^{-1} \circ \varphi(0,\, 1) = (0,\, 1)$ we obtain that $s_0 s'_0 = 1$ i.e. s_0 is invertible in k. In the same way $\varphi^{-1} \circ \varphi(a,\, 0) = (a,\, 0) = \varphi \circ \varphi^{-1}(a,\, 0)$ gives that ψ is bijective and $\psi' = \psi^{-1}$.

We will prove now the assertion from step (2). Assume that $\varphi : A_{(\lambda,\Lambda,\vartheta)} \to A^{(\lambda',u')}$ is an algebra map. Thus, φ is given by (32), for some quadruple $(s_0,\, \beta_0,\, \psi,\, r)$. Now, we can see that the algebra map condition $\varphi\big((0,x) \star (0,y)\big)= \varphi(0,x) \star' \varphi(0,y)$ holds if and only if $\beta_0 = 0$ and $s_0 = 0$, where $\star'$ denotes the multiplication on the algebra $A^{(\lambda',u')}$. Hence, φ takes the form $\varphi(a,\, x) = \big(\psi(a),\, r(a)\big)$, for all $a \in A$ and $x \in k$. Such a map is never an isomorphism of algebras since is not injective and thus two algebras of the form $A_{(\lambda,\Lambda,\vartheta)}$ and $A^{(\lambda',u')}$ are never isomorphic. The theorem is now completely proved. □

Remark 2.9. The compatibility condition (30) of Theorem 2.8 highlights the difficulty of classifying co-flag algebras over a given algebra A: it generalizes the classical Kroneker-Williamson equivalence of bilinear forms whose classification was started in [39] and finished in [27] over algebraically closed fields. We recall that two bilinear forms ϑ and ϑ' on a vector space A are called *isometric* if there exists a linear automorphism $\psi \in \mathrm{Aut}_k(A)$ such that $\vartheta(x, y) = \vartheta'(\psi(x),\, \psi(y))$, for all $x,\, y \in A$. If the cocycles ϑ and ϑ' are isometric as bilinear forms on A and ψ is an algebra automorphism of A, then

(30) holds by taking $s_0 := 1$ and $r := 0$, the trivial map. For future references to the problem of classifying bilinear forms up to an isometry we refer to [27].

Theorem 2.8 can be applied to classify all semidirect products of algebras of the form $A\#k$. We recall from Example 1.3 that a semidirect product $A\#k$ is just a Hochschild product $A_{(\lambda,\Lambda,\vartheta)} = A \star k$ having a trivial cocycle. The algebra obtained in this way will be denoted by $A_{(\lambda,\,\Lambda)}$. Directly from the proof of Theorem 2.8 we obtain:

Corollary 2.10. *Let A be an algebra, $(\lambda,\,\Lambda)$ and $(\lambda',\,\Lambda')$ two pairs consisting of algebra maps from A to k. Then there exists an isomorphism of algebras $A_{(\lambda,\,\Lambda)} \cong A_{(\lambda',\,\Lambda')}$ if and only if there exists $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(A)$ such that $\lambda = \lambda' \circ \psi$ and $\Lambda = \Lambda' \circ \psi$.*

An interesting special case occurs for the algebras A such that there is no algebra map $A \to k$ (e.g. the classical Weyl algebra $W_1(k) = k < x,\, y \,|\, xy - yx = 1 >$ or the matrix algebra $\mathrm{M}_n(k)$, for $n \geq 2$). Using Proposition 2.6 and Theorem 2.8 we obtain:

Corollary 2.11. *Let A be an algebra for which there is no algebra map $A \to k$. Then*

$$\mathbb{GH}^2\,(A,\,k) \cong k^*, \qquad \mathbb{HOC}\,(A,\,k) = \{\, A \times k \,\}$$

In particular, up to an isomorphism, the only algebra B for which there exists a surjective algebra map $B \to A$ having a 1-dimensional kernel is the direct product $A \times k$.

Determining the automorphism group of a given algebra is an old and very difficult problem, intensively studied in invariant theory (see [13] and their references). As already mentioned, the first step proved in the proof of Theorem 2.8 allows us to compute the automorphism group $\mathrm{Aut}_{\mathrm{Alg}}\,(A_{(\lambda,\Lambda,\vartheta)})$, for any $(\lambda,\Lambda,\vartheta) \in \mathcal{CF}_1\,(A)$. Let k^* be the units group of k, $k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)$ the direct product of groups and $(A^*,+)$ the underlying abelian group of the linear dual $A^* = \mathrm{Hom}_k(A,\,k)$. Then the map given for any $s_0 \in k^*$, $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(A)$ and $r \in A^*$ by:

$$\zeta : k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A) \to \mathrm{Aut}_{\mathrm{Gr}}\,(A^*,+), \qquad \zeta(s_0,\,\psi)\,(r) := s_0^{-1}\, r \circ \psi$$

is a morphism of groups. Thus, we can construct the semidirect product of groups $A^* \ltimes_\zeta \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big)$ associated to ζ. The next result shows that $\mathrm{Aut}_{\mathrm{Alg}}(A_{(\lambda,\Lambda,\vartheta)})$ is isomorphic to a certain subgroup of the semidirect product $A^* \ltimes_\zeta \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big)$.

Corollary 2.12. *Let A be an algebra, $(\lambda,\Lambda,\vartheta) \in \mathcal{CF}_1\,(A)$ a co-flag datum of the first kind of A and let $\mathcal{G}\big(A,\,(\lambda,\Lambda,\vartheta)\big)$ be the set of all triples $(s_0,\,\psi,\,r) \in k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A) \times A^*$ such that for any $a,\,b \in A$:*

$$\lambda = \lambda \circ \psi, \quad \Lambda = \Lambda \circ \psi, \quad \vartheta(a,b)\, s_0 = \vartheta\big(\psi(a),\psi(b)\big) + \lambda(a) r(b) + \Lambda(b) r(a) - r(ab)$$

Then, there exists an isomorphism of groups $\mathrm{Aut}_{\mathrm{Alg}}(A_{(\lambda,\Lambda,\vartheta)}) \cong \mathcal{G}\,\big(A,\,(\lambda,\Lambda,\vartheta)\big)$, where $\mathcal{G}\,\big(A,\,(\lambda,\Lambda,\vartheta)\big)$ is a group with respect to the following multiplication:

$$(s_0,\,\psi,\,r) \cdot (s_0',\,\psi',\,r') := (s_0 s_0',\,\psi \circ \psi',\,r \circ \psi' + s_0 r') \tag{36}$$

for all $(s_0,\,\psi,\,r)$, $(s_0',\,\psi',\,r') \in\in \mathcal{G}\,\big(A,\,(\lambda,\Lambda,\vartheta)\big)$. Moreover, the canonical map

$$\mathcal{G}\,\big(A,\,(\lambda,\Lambda,\vartheta)\big) \longrightarrow A^* \ltimes_\zeta \big(k^* \times \mathrm{Aut}_{\mathrm{Alg}}(A)\big), \qquad (s_0,\,\psi,\,r) \mapsto \big(s_0^{-1} r,\,(s_0,\,\psi)\big)$$

in an injective morphism of groups.

Proof. The fact that $\mathcal{G}\big(A,\, (\lambda, \Lambda, \vartheta)\big)$ is a group with respect to the multiplication (36) follows by a straightforward computation which is left to the reader: the unit is $(1,\, \mathrm{Id}_A,\, 0)$ and the inverse of $(s_0,\, \psi,\, r)$ is $(s_0^{-1},\, \psi^{-1},\, -s_0^{-1}(r \circ \psi^{-1}))$. The first statement follows from the proof of Theorem 2.8, step (1), since $\varphi_{(s_0, \psi, r)} \circ \varphi_{(s'_0, \psi', r')} = \varphi_{(s_0 s'_0,\, \psi \circ \psi',\, r \circ \psi' + s_0 r')}$, where $\varphi_{(s_0, \psi, r)}$ is an automorphism of $A_{(\lambda, \Lambda, \vartheta)}$ given by (31). The last assertion follows by a routine computation. □

Now we shall provide some explicit examples. The first example shows the limitations of the classical approach for the extension problem: there is no $(1+n^2)$-dimensional algebra with an algebra projection on the matrix algebra $\mathrm{M}_n(k)$ whose kernel is a null square ideal, but there exists a family of $(1+n^2)$-dimensional algebras which project on the matrix algebra $\mathrm{M}_n(k)$. We denote by $\{e_{ij} \,|\, i, j = 1, \cdots, n\}$ the canonical basis of $\mathrm{M}_n(k)$, i.e. e_{ij} is the matrix having 1 in the $(i,j)^{th}$ position and zeros elsewhere while δ_j^k and $\delta_{(n,n)}^{(i,j)}$ denote the Kroneker symbols.

Example 2.13. Let $n \geq 2$ be a positive integer. Then, $\mathbb{GH}^2\left(\mathrm{M}_n(k),\, k\right) \cong k^*$ and the equivalence classes of all $(1+n^2)$-dimensional algebras with an algebra projection on $\mathrm{M}_n(k)$ are the following algebras denoted by $\mathrm{M}_n(k)^u$ and defined for any $u \in k^*$ as the vector space having $\{f,\, e_{ij} \,|\, i, j = 1, \cdots, n\}$ as a basis and the multiplication given for any $i,\, j = 1, \cdots, n$ by:

$$f^2 := u\, f, \quad e_{ij} \star f = f \star e_{ij} := \delta_{(n,n)}^{(i,j)}\, f, \quad e_{ij} \star e_{kl} := \delta_k^j\, e_{il} + u^{-1}\big(\delta_{(n,n)}^{(i,j)} \delta_{(n,n)}^{(k,l)} - \delta_j^k\, \delta_{(n,n)}^{(i,l)}\big)\, f$$

Furthermore, $\mathbb{HOC}\left(\mathrm{M}_n(k),\, k\right) = \{\mathrm{M}_n(k) \times k\}$.

The result follows by applying Proposition 2.6 and Corollary 2.11 since there is no unitary algebra map $\mathrm{M}_n(k) \to k$. If we consider $\lambda^0 : \mathrm{M}_n(k) \to k$ defined by $\lambda^0(e_{ij}) := \delta_{(n,n)}^{(i,j)}$, for all $i,\, j = 1, \cdots, n$, then $\{(\lambda^0,\, u) \,|\, u \in k^*\}$ is a system of representatives for the equivalence relation $\approx_2$. The algebra $\mathrm{M}_n(k)^u$ associated to the pair $(\lambda^0,\, u)$ is the vector space having $\{f,\, e_{ij} \,|\, i, j = 1, \cdots, n\}$ as a basis while the multiplication given by (23) comes down to the one in the statement.

An interesting example through the subtle arithmetics involved in the classification of the corresponding Hochschild products is the group algebra $k[C_n]$, where for a positive integer $n \geq 2$ we denote by C_n the cyclic group of order n generated by d. We introduce the following notation: for any $i,\, j = 1, \cdots, n-1$ we shall denote by $i * j$ the positive integer given by

$$i * j := \begin{cases} i + j & \text{if} \quad j + i < n \\ i + j - n & \text{if} \quad j + i \geq n \end{cases}$$

In what follows $U_n(k) := \{\omega \in k \mid \omega^n = 1\}$ denotes the cyclic group of n-th roots of unity in k and $\mathcal{A}(n,\, k) := \{x \in U(k[C_n]) \mid \psi : k[C_n] \to k[C_n],\, \psi(d^i) = x^i,\, i = 0, 1, \cdots, n-1, \text{ is an algebra automorphism}\}$.

Example 2.14. Let k be a field such that n is invertible in k. Then:

$$\mathbb{GH}^2\left(k[C_n],\, k\right) \cong \big(U_n(k) \times U_n(k)\big) \sqcup k^*$$

and the equivalence classes of $(n+1)$-dimensional algebras with an algebra projection on $k[C_n]$ are the families of algebras having $\{f,\, d^i \mid i = 1, \cdots, n\}$ as a basis over k and the multiplication $\star$ defined for any $(\alpha,\, \beta) \in U_n(k) \times U_n(k)$, $u \in k^*$ and $i,\, j = 1, \cdots, n$ by:

$$\begin{aligned} k[C_n]_{(\alpha,\,\beta)}: &\qquad d^i \star d^j = d^{i+j}, \quad f^2 = 0, \quad d^i \star f = \alpha^i\, f, \quad f \star d^i = \beta^i\, f \\ k[C_n]^u: &\qquad d^i \star d^j = d^{i+j} + u^{-1}(\delta_i^n\, \delta_j^n - \delta_{i+j}^n) f, \quad f^2 = uf, \quad d^i \star f = f \star d^i = \delta_i^n\, f \end{aligned}$$

Furthermore, there exists a bijection

$$\mathbb{HOC}\,(k[C_n],\, k) \cong \big(U_n(k) \times U_n(k)/ \equiv\big) \sqcup \{k[C_n] \times k\}$$

where $\equiv$ is the following equivalence relation on $U_n(k) \times U_n(k)$: two pairs $(\alpha,\, \beta)$, $(\alpha',\, \beta')$ of n-th roots of unity in k are equivalent $(\alpha,\, \beta) \equiv (\alpha',\, \beta')$ if and only if there exists $x_0 + x_1\, d + \cdots + x_{n-1}\, d^{n-1} \in \mathcal{A}(n,\, k)$ such that

$$\alpha' = x_0 + x_1\, \alpha + \cdots + x_{n-1}\, \alpha^{n-1}, \qquad \beta' = x_0 + x_1\, \beta + \cdots + x_{n-1}\, \beta^{n-1} \tag{37}$$

To start with we point out that the algebra maps $k[C_n] \to k$ are parameterized by the cyclic group of n-th roots of unity in k. Consider $\alpha,\, \beta \in U_n(k)$ which implement λ and respectively Λ, i.e. $\lambda(d) = \alpha$ and $\Lambda(d) = \beta$. We are left to compute the set of all $(\lambda,\, \Lambda)$-cocycles. To this end we denote $\vartheta(d^i,\, d) := \xi_i$, $i = 1, \cdots, n-1$ and we will see that these elements will allow us to completely determine the cocycle $\vartheta : k[C_n] \times k[C_n] \to k$. Indeed, by writing down (17) for triples of the form $(d^i,\, d^j,\, d)$ and using induction we obtain the following general formula:

$$\vartheta(d^i,\, d^j) = \sum_{k=0}^{j-1} \xi_{i*k}\, \beta^{j-1-k} - \big(\sum_{l=1}^{j-1} \xi_l\, \beta^{j-1-l}\big)\alpha^i$$

for all $i,\, j = 1, \cdots, n-1$, where $\xi_0 := 0$. Furthermore, by writing down (17) for triples of the form $(d^i,\, d^{n-i},\, d^i)$ and using the above formula for ϑ we obtain the following compatibility which needs to be fulfilled for any $i = 1, \cdots, n-1$:

$$(\alpha^i - \beta^i)(\xi_{n-1} + \xi_{n-2}\, \beta + ... + \xi_1\, \beta^{n-2}) = 0$$

Therefore we distinguish two cases, namely: $\alpha = \beta$ or $\alpha \neq \beta$ and $\xi_{n-1} + \xi_{n-2}\, \beta + ... + \xi_1\, \beta^{n-2} = 0$. It follows that $\mathcal{CF}_1\,(k[C_n]) \cong (U_n(k) \times k^{n-1}) \cup \{(\alpha,\, \beta,\, \xi_1,\, \xi_2,\, ...,\, \xi_{n-2}) \in U_n(k) \times U_n(k) \times k^{n-2} \mid \alpha \neq \beta\}$ and the bijection associates to any $(\alpha,\, \delta_1,\, \delta_2,\, ...,\, \delta_{n-1}) \in U_n(k) \times k^{n-1}$ the co-flag datum of the first kind $(\lambda_\alpha,\, \Lambda_\alpha,\, \vartheta_\delta)$ given for all $i,\, j = 1, \cdots, n-1$ by:

$$\lambda(d) = \Lambda(d) := \alpha, \;\; \vartheta_\delta(d^i,\, d^j) := \sum_{k=0}^{j-1} \delta_{i*k}\, \alpha^{j-1-k} - \big(\sum_{l=1}^{j-1} \delta_l\, \alpha^{i+j-1-l}\big)$$

where $\delta_0 = 0$, and to any $(\beta,\, \gamma,\, \xi_1,\, \xi_2,\, ...,\, \xi_{n-2}) \in U_n(k) \times U_n(k) \times k^{n-2}$, with $\beta \neq \gamma$, associates the co-flag datum of the first kind $(\overline{\lambda}_\beta,\, \overline{\Lambda}_\gamma,\, \overline{\vartheta}_\xi)$ given for any $i,\, j = 1, \cdots, n-1$ by:

$$\overline{\lambda}_\beta(d) := \beta, \;\; \overline{\Lambda}_\gamma(d) := \gamma, \;\; \overline{\vartheta}_\xi(d^i,\, d^j) := \sum_{k=0}^{j-1} \xi_{i*k}\, \gamma^{j-1-k} - \big(\sum_{l=1}^{j-1} \xi_l\, \gamma^{j-1-l}\big)\beta^i$$

where $\xi_0 = 0$ and $\xi_{n-1} = -\big(\xi_{n-2}\,\beta + \,...\, + \xi_1\,\beta^{n-2}\big)$. It is now obvious that a $(\lambda_\alpha,\, \Lambda_\alpha)$-cocycle is never equivalent to a $(\overline{\lambda}_\beta,\, \overline{\Lambda}_\gamma)$-cocycle for any $\alpha,\, \beta,\, \gamma \in U_n(k)$, $\beta \neq \gamma$. Furthermore, by a rather long but straightforward computation it can be easily seen that for all $\alpha \in U_n(k)$, any $(\lambda_\alpha,\, \Lambda_\alpha)$-cocycle, say ϑ_δ, is equivalent (in the sense of (26)) to the trivial cocycle via the linear map $t : k[C_n] \to k$ defined by $t(1) := 0$ and for any $i = 2, \cdots, n-1$:

$$t(d) := n^{-1} \sum_{j=1}^{n-1} \alpha^j \delta_{n-j}, \quad t(d^i) := n^{-1} i \alpha^{i-1} \sum_{j=1}^{n-1} \alpha^j \delta_{n-j} - \sum_{j=0}^{i-2} \alpha^j \delta_{i-1-j}$$

Therefore we have $\sqcup_{(\lambda_\alpha, \Lambda_\alpha)}\, \mathrm{H}^2_{(\lambda_\alpha,\, \Lambda_\alpha)}\,(k[C_n],\, k) \cong \{(\alpha,\, \alpha) \mid \alpha \in U_n(k)\}$. A similar statement holds for the second family of co-flag data of the first kind: for all $\beta,\, \gamma \in U_n(k)$, with $\beta \neq \gamma$, any $(\overline{\lambda}_\beta,\, \overline{\Lambda}_\gamma)$-cocycle, say $\overline{\vartheta}_\xi$, is equivalent to the trivial cocycle via the linear map $\overline{t} : k[C_n] \to k$ defined by $\overline{t}(1) := \overline{t}(d) := 0$ and $\overline{t}(d^i) := -\sum_{j=0}^{i-2} \xi_{i-1-j}\, \gamma^j$, for all $i = 2, \cdots, n-1$. Thus $\sqcup_{(\overline{\lambda}_\beta, \overline{\Lambda}_\gamma)}\, \mathrm{H}^2_{(\overline{\lambda}_\beta,\, \overline{\Lambda}_\gamma)}\,(k[C_n],\, k) \cong \{(\alpha,\, \beta) \in U_n(k) \times U_n(k) \mid \alpha \neq \beta\}$.

Therefore, we have proved that $\Big(\mathcal{CF}_1\,(k[C_n])/\approx_1\Big) \cong U_n(k) \times U_n(k)$ and the corresponding co-flag algebras are those denoted by $k[C_n]_{(\alpha,\, \beta)}$. For the co-flag data of the second kind of $k[C_n]$ we choose the set of representatives $\{(\lambda^0, u) \mid u \in k^*\}$ for the equivalence relation $\approx_2$, where $\lambda^0 : k[C_n] \to k$ is given by $\lambda^0(d^i) = \delta_i^n$, for all $i = 1, \cdots, n$. They give rise to the co-flag algebras denoted by $k[C_n]^u$. The conclusion now follows from Corollary 2.7. Finally, the assertion regarding $\mathbb{HOC}\,(k[C_n],\, k)$ is an easy consequence of Corollary 2.10.

Remark 2.15. Example 2.14 shows that any Hochschild product $k[C_n] \star k$ is isomorphic either to the direct product $k[C_n] \times k$, or to a semi-direct product $k[C_n]_{(\alpha,\, \beta)}$, parameterized by the group $U_n(k) \times U_n(k)$. The explicit description of the isomorphism classes of the algebras $k[C_n]_{(\alpha,\, \beta)}$ indicated by the equivalence relation (37) is a difficult number theory problem which relies heavily on the arithmetics of the positive integer n as well as on the base field k. Furthermore, the problem is also related to other two intensively studied problems in the theory of group algebras, namely the description of all invertible elements and the automorphism group of a group algebra [23, 32, 34]. Indeed, the key set $\mathcal{A}(n,\, k)$ which appears in the description of the classifying object $\mathbb{HOC}\,(k[C_n],\, k)$ parameterizes in fact the automorphism group $\mathrm{Aut}_{\mathrm{Alg}}(k[C_n])$. Any element of $\mathcal{A}(n,\, k)$ is invertible in $k[C_n]$ and has order n in the group $U(k[C_n])$. These elements depend essentially on n and the base field k. Indeed, let $X^n - 1 = f_1 f_2 \cdots f_t$ be the decomposition of $X^n - 1$ as a product of irreducible polynomials in $k[X]$. If we denote by ε_i a root of f_i in a fixed algebraic closure of k, we obtain that there exists a canonical isomorphism of k-algebras $k[C_n] \cong k(\varepsilon_1) \times \cdots \times k(\varepsilon_t)$ that maps the generator d of C_n to $(\varepsilon_1, \cdots, \varepsilon_t)$. Thus, $\mathrm{Aut}_{\mathrm{Alg}}(k[C_n])$ is isomorphic to a direct product between all wreath product of $\mathrm{Aut}\big(k(\varepsilon_i)\big)$ and the symmetric groups [34].

Applying Example 2.14 for $n = 2$, we obtain the classification of all 3-dimensional algebras with an algebra projection on $k[C_2] \cong k \times k$.

Example 2.16. If k a field of characteristic $\neq 2$, then:

$$\mathbb{GH}^2\left(k[C_2],\, k\right) \cong \big(\{\pm 1\} \times \{\pm 1\}\big) \sqcup k^* \tag{38}$$

$$\mathbb{HOC}\left(k[C_2],\, k\right) \cong \{k^3,\ k[X,Y]/(X^2-1,\, Y^2,\, XY-Y),\ A_{21}\} \tag{39}$$

where A_{21} is the 3-dimensional non-commutative algebra having $\{1, d, f\}$ as a basis and the multiplication given by $d^2 = 1,\ f^2 = 0,\ df = -fd = f$.

Now we highlight the efficiency of our methods in order to classify co-flag algebras of a given dimension. If k is a field of characteristic $\neq 2$, then, up to an isomorphism, there exists only two co-flag algebras of dimension 2: the algebras $k[X]/(X^2)$ and $k[C_2] \cong k \times k$ [5, Corollary 4.5]. If $k \neq k^2$, we mention that the other family of 2-dimensional algebras, namely the quadratic field extension $k(\sqrt{d})$, for some $d \in k \setminus k^2$ does not contain co-flag algebras since there is no algebra map $k(\sqrt{d}) \to k$. The co-flag algebras over $k[C_2]$ are classified by (39) and thus, in order to classify all 3-dimensional co-flag algebras we need to classify the co-flag algebras over $k[X]/(X^2)$.

Example 2.17. Let $A := k[X]/(X^2)$. Then $\mathbb{GH}^2\left(k[X]/(X^2),\, k\right) \cong k \sqcup k^*$ and the equivalence classes of 3-dimensional algebras that have an algebra projection on $k[X]/(X^2)$ are two families of algebras defined for any $a \in k$ and $u \in k^*$ as follows:

$$A_a := k[X,\, Y]/(X^2 - a\,Y,\, Y^2,\, XY), \qquad A^u := k[X,\, Y]/(X^2,\, Y^2 - u\,Y,\, XY)$$

Furthermore, $\mathbb{HOC}\left(k[X]/(X^2),\, k\right) = \{A_0,\, A_1,\, A^1\}$, i.e. up to an isomorphism there exist three co-flag algebras of dimension 3 over $k[X]/(X^2)$.

Indeed, A is the 2-dimensional algebra having 1 and x as a basis and $x^2 = 0$. Thus A has only one algebra map $A \to k$, namely the one sending x to 0. Hence, there exists a bijection $\mathcal{CF}_1(A) \cong k$ such that the co-flag datum of the first kind $(\lambda, \Lambda, \vartheta)$ associated to $a \in k$ is given by

$$\vartheta(x,x) := a, \quad \lambda(x) = \Lambda(x) = \vartheta(1,x) = \vartheta(x,1) = \vartheta(1,1) := 0$$

We can easily see that the equivalence relation $\approx_1$ of Proposition 2.6 becomes equality, i.e. $a \approx_1 a'$ if and only if $a = a'$ and hence $\mathcal{CF}_1(A)/\approx_1 \cong k$. The families of algebras associated to such a co-flag datum of the first kind as defined by (22) are the 3-dimensional algebras having $\{f,\, 1,\, x\}$ as a basis and the multiplication given by: $x \star x = af$, $f^2 = x \star f = f \star x = 0$, which is the algebra A_a. For the last part we apply Proposition 2.6 which proves that $\mathcal{CF}_2(A)/\approx_2 \cong k^*$: the algebra A^u, for all $u \in k^*$, is precisely the algebra defined by (23) associated to the co-flag datum of the second king (λ^0, u), where λ^0 is the linear map given by $\lambda^0(x) := 0$, $\lambda^0(1) := 1$. The last statement follows from Theorem 2.8 or it can be proved directly as follows: we observe that, for any $u \in k^*$, there exists and isomorphism of algebras $A^u \cong A^1 = k[X,\, Y]/(X^2,\, Y^2 - Y,\, XY)$. On the other hand, there exists an isomorphism of algebras $A_a \cong A_1 = k[X,\, Y]/(X^2 - Y,\, Y^2,\, XY)$, for all $a \in k^*$ and any two algebras A_0, A_1 and A^1 are not isomorphic to each other.

To conclude, using Example 2.16 and Example 2.17 we obtain:

Corollary 2.18. *If k is a field of characteristic $\neq 2$ then, up to an isomorphism, there exist exactly six 3-dimensional co-flag algebras namely:*

$$k^3, \quad k[X,Y]/(X^2-1, Y^2, XY-Y), \quad k< x,\, y\,|\, x^2=1,\, y^2=0,\, xy=-yx=y>$$
$$k[X,\, Y]/(X^2,\, Y^2,\, XY), \;\; k[X,\, Y]/(X^2-Y,\, Y^2,\, XY), \;\; k[X,\, Y]/(X^2,\, Y^2-Y,\, XY)$$

In particular, if $k := \mathbb{C}$ the field of complex numbers, Corollary 2.18 shows that only 6 out of the 22 types of algebras of dimension 3 are co-flag algebras. Moreover, we also highlight the efficiency of Theorem 2.8 in classifying co-flag algebras by turning the problem into a purely computational one using a recursive method: if we consider A to be each of the algebras from Corollary 2.18 and using the results of this section we will arrive at the classification of 4-dimensional co-flag algebras. Of course the difficulty of the computations increases along side with the dimension.

Very interesting and completely different from $\mathrm{M}_n(k)$ is the case when $A := \mathcal{T}_n(k)$ is the algebra of upper triangular matrices, i.e. $\mathcal{T}_n(k)$ is the subalgebra of $\mathrm{M}_n(k)$ having $B := \{e_{ij} \mid i,\, j = 1,\, 2,\, \cdots,\, n,\, i \leq j\}$ as the canonical basis over k. In order to write down the classifying object $\mathbb{GH}^2\,(\mathcal{T}_n(k),\, k)$ we introduce the following three sets of matrices of trace zero, defined for any $u,\, v,\, w = 1, 2, \cdots, n$ by:

$$\begin{aligned}
M^u \;\; &:= \;\; \{A = (a_{ij}) \in \mathrm{M}_n(k) \mid a_{ii} = 0, \text{ for all } i = 1, \cdots, n \text{ and } a_{ur} = 0, \text{ for all } u < r\}\\
M^{v,w} \;\; &:= \;\; \{A = (a_{ij}) \in \mathrm{M}_n(k) \mid \sum_{i=1}^{n} a_{ii} = 0,\;\; a_{vv} = 0, \text{ and } a_{ws} = 0, \text{ for all } w \leq s\}\\
\overline{M}^{v,w} \;\; &:= \;\; \{A = (a_{ij}) \in \mathrm{M}_n(k) \mid \sum_{i=1}^{n} a_{ii} = 0,\;\; a_{vv} = 0, \text{ and } a_{ws} = 0, \text{ for all } w \leq s \neq v\}
\end{aligned}$$

Example 2.19. Let k be a field of characteristic zero. Then:

$$\mathbb{GH}^2\,(\mathcal{T}_n(k),\, k) \cong \big(\bigcup_{u \in \{1,2,\cdots,n\}} M^u\big) \sqcup \big(\bigcup_{v,w \in \{1,2,\cdots,n\},\, v<w} M^{v,w} \times k^{n-w}\big) \sqcup U \sqcup k^*$$

where we denote: $U := \bigcup_{v,w \in \{1,2,\cdots,n\},\, v>w} \overline{M}^{v,w} \times k^{n-w} \times k^{v-w-1}$. The equivalence classes of $\big(\frac{n(n+1)}{2}+1\big)$-dimensional algebras that have an algebra projection on $\mathcal{T}_n(k)$ are the families of algebras having $\{f,\, e_{ij} \mid i,\, j = 1, 2, \cdots, n,\, i \leq j\}$ as a basis over k and the multiplication given below (we only write down the non-zero products):

$$\begin{aligned}
\mathcal{T}_n(k)^u_A: \quad & e_{it} \star e_{ts} = e_{il} - \alpha_{is} f,\;\; e_{uj} \star e_{jl} = e_{ul},\;\; e_{uu} \star e_{kl} = \alpha_{kl} f,\;\; e_{ij} \star e_{uu} = \alpha_{ij} f,\\
& e_{ij} \star e_{jj} = e_{ij} - \alpha_{ij} f,\;\; e_{iu} \star e_{uu} = e_{iu},\;\; e_{uu} \star f = f \star e_{uu} = f,\\
& \text{where } u \in \{1, 2, \cdots, n\},\; A = (\alpha_{pq})_{p,q=\overline{1,n}} \in M^u,\; i, j \neq u,\; t \neq s;\\
\mathcal{T}_n(k)^{v,w}_{B,\Gamma}: \quad & e_{ip} \star e_{pl} = e_{il} - \beta_{il} f,\;\; e_{wi} \star e_{il} = e_{wl} - \beta_{l} f,\;\; e_{ww} \star e_{kl} = e_{wl}\delta^w_k + \beta_{kl} f,\\
& e_{it} \star e_{vv} = \beta_{it} f,\;\; e_{ij} \star e_{jj} = e_{ij} - \beta_{ij}(1 - \delta^v_j) f,\;\; e_{ws} \star e_{vv} = -\gamma_s f,\\
& e_{wj} \star e_{jj} = e_{wj} + \gamma_j (1 - \delta^w_j) f,\;\; e_{vv} \star f = e_{ww} \star f = f,\\
& \text{where } v, w \in \{1, 2, \cdots, n\},\; v < w,\; B = (\beta_{pq})_{p,q=\overline{1,n}} \in M^{v,w},\\
& \Gamma = (\gamma_r)_{w<r},\; i \neq w,\; p \neq l,\; i \neq l,\; t \neq v,\; s \notin \{v, w\};
\end{aligned}$$

$$\begin{array}{ll}\mathcal{T}_n(k)^{w,v}_{C,\Psi,\Omega}: & e_{ip}\star e_{pl}=e_{il}-\delta_{il}f,\ e_{wi}\star e_{ij}=e_{wj}+\psi_j f,\ e_{wt}\star e_{tv}=e_{wv}+\omega_t f,\\ & e_{ww}\star e_{kl}=e_{wl}\delta^w_k+\delta_{kl}f,\ e_{ij}\star e_{jj}=e_{ij}-\delta_{ij}f,\ e_{wj}\star e_{vv}=-\psi_j f\\ & e_{iv}\star e_{vv}=e_{iv},\ e_{vv}\star f=f\star e_{ww}=f,\ \text{where}\, v,w\in\{1,2,\cdots,n\},\\ & v>w,\ C=(\delta_{pq})_{p,q=\overline{1,n}}\in\overline{M}^{v,w},\ \Psi=(\psi_r)_{w<r},\ \Omega=(\omega_s)_{w<s<v},\\ & i\neq w,\ j\neq v,\ p\neq l,\ t\notin\{v,w\};\\ {}^{\lambda}\mathcal{T}_n(k): & e_{ij}\star e_{jl}=e_{il}+\lambda^{-1}\delta^n_i\delta^n_l(\delta^n_j-1)f,\ e_{nn}\star f=f\star e_{nn}=f,\ f^2=\lambda f,\\ & \text{where}\,\lambda\in k^*.\end{array}$$

Indeed, we start by discussing the algebra maps $\lambda:\mathcal{T}_n(k)\to k$. Denote $\lambda(e_{ij})=\alpha_{ij}\in k$, for all $e_{ij}\in B$. Since $e_{ii}^2=e_{ii}$ we have $\alpha_{ii}\in\{0,1\}$, for all $i=1,\cdots,n$. Moreover, since $\mathrm{char}(k)=0$ and we assume λ to be unitary it follows that $\Sigma_{i=1}^n\alpha_{ii}=1$. Therefore, $\alpha_{uu}=1$, for some $u\in\{1,\cdots,n\}$ and $\alpha_{ii}=0$, for all $i\neq u$; we denote by λ^u this algebra map. As $e_{ij}=e_{ii}e_{ij}$, for all $i\leq j$, we obtain that $\alpha_{ij}=0$, for all $i\neq u$ and $i\leq j$. Finally, since $e_{uj}e_{uu}=0$ and $\lambda^u(e_{uu})=1$, we obtain that $\alpha_{uj}=0$, for any $u<j$. To conclude, the set of algebra maps $\mathcal{T}_n(k)\to k$ are in bijection to the set $\{1,\cdots,n\}$ and the algebra map corresponding to some $j\in\{1,\cdots,n\}$ is given by $\lambda^j(e_{jj}):=1$ and $\lambda^j(e_{uv}):=0$, for all $(u,v)\neq(j,j)$. The next step of the proof is a computational one: namely, for any $u,\ v\in\{1,\ 2,\ \cdots,\ n\}$ we are left to compute the set of all $(\lambda^v,\ \Lambda^u)$-cocycles ϑ. This is achieved by straightforward but lengthy checking of (17) which in this case comes down to the following compatibility condition:

$$\vartheta(e_{ij},\ e_{rs}e_{pq})-\vartheta(e_{ij}e_{rs},\ e_{pq})=\vartheta(e_{ij},\ e_{rs})\Lambda^u-\vartheta(e_{rs},\ e_{pq})\lambda^v \tag{40}$$

with $i\leq j$, $r\leq s$ and $p\leq q$. Rather than including here the cumbersome computations we will just point out the main steps taken; the detailed proof can be provided upon request. First, notice that since (40) is not symmetric with respect to the maps λ^v and Λ^u we distinguish three cases, namely: $u=v$, $u<v$ and respectively $u>v$. For the case $u=v$ the $(\lambda^u,\ \Lambda^u)$-cocycles obtained are implemented by a family of $(n-u)$ scalars and a matrix of trace zero with zeros on the line u strictly above the diagonal. Then, it can be proved that any such cocycle is equivalent (in the sense of (26)) to a cocycle implemented by a matrix in M^u. The corresponding co-flag algebras are those denoted by $\mathcal{T}_n(k)^u_A$. If $u<v$ then any $(\lambda^v,\ \Lambda^u)$-cocycle is equivalent to a cocycle implemented by $(n-v)$ scalars and a matrix in $M^{u,v}$. The corresponding co-flag algebras are those denoted by $\mathcal{T}_n(k)^{u,v}_{B,\Gamma}$. Finally, $u>v$ then any $(\lambda^v,\ \Lambda^u)$-cocycle is equivalent to a cocycle implemented by two families of $(n-v)$ and respectively $(u-v-1)$ scalars and a matrix in $\overline{M}^{u,v}$. The corresponding co-flag algebras are those denoted by $\mathcal{T}_n(k)^{v,u}_{C,\Psi,\Omega}$. Finally, the last family of co-flag algebras, denoted by ${}^{\lambda}\mathcal{T}_n(k)$, corresponds to a co-flag datum of the second kind associated $(\delta^n,\ \lambda)$, where $\delta^n(i)=\delta^n_i$ is the Kronecker symbol and $\lambda\in k^*$.

Moreover, for $n=2$ we can also write down in a transparent way the other classifying object, namely $\mathbb{HOC}\,(\mathcal{T}_2(k),\ k)$. By a long but straightforward computation it can easily be seen that $\mathbb{HOC}\,(\mathcal{T}_2(k),\ k)$ contains the algebras whose multiplication is depicted below together with the direct product of algebras $\mathcal{T}_2(k)\times k$:

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	e_{12}	0	f
e_{12}	0	0	e_{12}	0
e_{22}	0	0	e_{22}	0
f	f	0	0	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	e_{12}	0	0
e_{12}	0	0	e_{12}	0
e_{22}	0	0	e_{22}	f
f	0	0	f	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	$e_{11}-f$	e_{12}	f	0
e_{12}	0	0	e_{12}	0
e_{22}	$-f$	0	$e_{22}-f$	f
f	0	0	f	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	e_{12}	0	f
e_{12}	0	0	e_{12}	0
e_{22}	0	0	e_{22}	0
f	0	0	f	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	$e_{12}-f$	0	f
e_{12}	f	0	$e_{12}-f$	0
e_{22}	0	f	e_{22}	0
f	0	0	f	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	e_{12}	0	0
e_{12}	0	0	e_{12}	0
e_{22}	0	0	e_{22}	f
f	f	0	0	0

$\star$	e_{11}	e_{12}	e_{22}	f
e_{11}	e_{11}	$e_{12}+f$	0	0
e_{12}	0	0	e_{12}	0
e_{22}	0	0	e_{22}	f
f	f	0	0	0

3. Applications to coalgebras and Poisson algebras

In this section we shall present two applications of our results to the theory of coalgebras and respectively Poisson algebras, the latter being the algebraic counterpart of Poisson manifolds. Our strategy is to use two different contravariant functors which have both the category of algebras as a codomain, namely the linear dual functor $(-)^* := \mathrm{Hom}_k\,(-,\, k)$ and respectively $\mathrm{Fun}\,(-) := C^\infty(-)$ the real smooth functions on a Poisson manifold functor.

Supersolvable coalgebras. We recall that a coalgebra $C = (C,\, \Delta,\, \varepsilon)$ is a vector space C equipped with a comultiplication $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to k$ such that $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$ and $(\mathrm{Id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}$, where $\otimes = \otimes_k$ and Id is the identity map on C. We use the Σ-notation for comultiplication: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for all $c \in C$ (summation understood). The base field k, with the obvious structures, is the final object in the category of coalgebras. We only provide some basic information of coalgebras, referring the reader to [11] for more detail. If $C = (C,\, \Delta,\, \varepsilon)$ is a colgebra, then the linear dual $C^* = \mathrm{Hom}_k(C,\, k)$ is an associative algebra under the convolution product $(f * g)(c) := f(c_{(1)})g(c_{(2)})$, for all $f,\, g \in C^*$ and $c \in C$ having the unit $1_{C^*} = \varepsilon$. Conversely, if A is a finite dimensional algebra with a basis $\{e_i \,|\, i = 1, \cdots, n\}$, then the linear dual A^* is a coalgebra with the comultiplication and the counit given for any $i = 1, \cdots, n$ by:

$$\Delta_{A^*}(e_i^*) := \sum_{j,l=1}^{n} e_i^*(e_j e_l)\, e_j^* \otimes e_l^*, \qquad \varepsilon_{A^*}(e_i^*) := e_i^*(1_A) \tag{41}$$

where $\{e_i^* \,|\, i = 1, \cdots, n\}$ is the dual basis of $\{e_i \,|\, i = 1, \cdots, n\}$. The contravariant functor $(-)^* := \mathrm{Hom}_k\,(-,\, k)$ gives a duality between the category of all finite dimensional coalgebras and the category of finite dimensional algebras [11]. Having the supersolvable Lie algebras [9] as a source of inspiration we introduce the following concept:

Definition 3.1. A coalgebra C is called *supersolvable* if there exists a positive integer n such that C has a finite chain of subcoalgebras

$$k \cong C_1 \subset C_2 \subset \cdots \subset C_n := C \tag{42}$$

such that C_i has codimension 1 in C_{i+1}, for all $i = 1, \cdots, n-1$.

Any supersolvable coalgebra is finite dimensional and the definition is equivalent to the fact that $\dim(C_i) = i$, for all $i = 1, \cdots, n$. Furthermore, since any supersolvable coalgebra C contains a 1-dimensional subcolagebra, we obtain that $G(C) \neq \emptyset$, where $G(C)$ is the space of group-like elements of C. Using the duality given by the functor $(-)^*$ it follows that a finite dimensional coalgebra C is supersolvable if and only if its convolution algebra C^* is a co-flag algebra in the sense of Definition 2.1 and vice-versa: a finite dimensional algebra A is a co-flag algebra if and only if its dual A^* is a supersolvable coalgebra. Thus, the results obtained in the previous section can be applied for the classification problem of supersolvable coalgebras of a given dimension.[2] Using the above facts, Corollary 2.18 classifies in fact all 3-dimensional supersolvable coalgebras: the isomorphism classes are the duals A^* of the algebras listed in the statement. For example, the dual coalgebra associated to the noncommutative algebra $k < x, y \,|\, x^2 = 1,\, y^2 = 0,\, xy = -yx = y >$ is the non-cocommutative supersolvable coalgebra having $\{f_1, f_2, f_3\}$ as a basis and the comultiplication and the counit given by:

$$\Delta(f_1) := f_1 \otimes f_1 + f_2 \otimes f_2, \quad \Delta(f_2) := f_1 \otimes f_2 + f_2 \otimes f_1, \quad \varepsilon(f_1) := 1, \quad \varepsilon(f_2) := 0$$
$$\Delta(f_3) := f_1 \otimes f_3 + f_3 \otimes f_1 + f_2 \otimes f_3 - f_3 \otimes f_2, \qquad \varepsilon(f_3) := 0$$

The coalgebra obtained in this way is indeed supersolvable by choosing $C_1 := k(f_1 - f_2)$ (we observe that $f_1 \pm f_2$ is a group-like element) and $C_2 := kf_1 + kf_2$ as the intermediary coalgebras of dimension 1 respectively 2 in the sequence (42).

Another application of our theory of Section 1 is the following: let C be a finite dimensional coalgebra, n a positive integer and $V := k^n$. Then by taking the convolution algebra $A := C^*$ we obtain that the object $\mathbb{GH}^2\,(C^*,\, k^n)$ classifies, up to an isomorphism which stabilizes C and costabilizes k^n, all coalgebras which contain C as a subcoalgebra of codimension n. Moreover, a coalgebra D contains C as a subcoalgebra of codimension n if and only if D is isomorphic to the dual coalgebra $(C^* \star k^n)^*$ of a Hochschild product $C^* \star k^n$ between the convolution algebra C^* and the vector space k^n. The formula for the comultiplication of any such coalgebra $(C^* \star k^n)^*$ can be written down effectively by using (41) and (3). This observation shows that the GE-problem applied for finite dimensional algebras and finite dimensional vector spaces gives the answer at the level of finite dimensional coalgebras to what we have called the *extending structures problem* studied in [2, 4, 5] for Jacobi, Lie and respectively associative algebras.

Applications to Poisson algebras. Commutative Poisson algebras are algebraic counterparts of Poisson manifolds from differential geometry: for a given smooth manifold M, there is a one-to-one correspondence between Poisson brackets on the commutative algebra $P := C^\infty(M)$ of smooth functions on M and all Poisson structures on M [22]. The

[2] The classification of solvable Lie algebras [20], over arbitrary fields, was achieved up to dimension 4.

importance of Poisson algebras in several areas of mathematics and physics (Hamiltonian mechanics, differential geometry, Lie groups, noncommutative algebraic/diferential geometry, (super)integrable systems, quantum field theory, vertex operator algebras, quantum groups) it is also well known – see [21, 22, 30] and the references therein. In fact, $C^{\infty}(-)$ gives a contravariant functor from the category of Poisson manifolds to the category of Poisson algebras and constitutes the tool through which geometrical problems can be restated and approached at the level of Poisson algebras. In particular, the GE-problem, formulated for Poisson algebras, is just the algebraic counterpart of the following question from differential geometry: *Let M be a Poisson manifold. Describe and classify all Poisson manifolds containing M as a manifold of a given codimension.*

We recall that a *Poisson algebra* is a triple $P = (P, m_P, [-, -])$, where (P, m_P) is a unital associative algebra, $(P, [-, -])$ is a Lie algebra such that the Leibniz law holds for any $p, q, r \in P$:

$$[pq, r] = [p, r]\, q + p\, [q, r] \tag{43}$$

Usually, a Poisson algebra P is by definition assumed to be commutative like the algebra $P := C^{\infty}(M)$ of real smooth functions on a Poisson manifold M which is the typical example of a Poisson algebra. However, following [18, 29] in order to broaden the class of Poisson algebras and to be able to construct relevant examples, throughout this paper we do not impose this restriction. A morphism between two Poisson algebras P and P' is a linear map $\varphi : P \to P'$ that is a morphism of associative algebras as well as of Lie algebras; we will denote by $\mathrm{Aut}_{\mathrm{Poss}}(P)$ the group of automorphisms of a Poisson algebra P. For basic concepts and unexplained notions on Lie algebras we refer to [16] and to [30] for those concerning Poisson algebras.

Let $P = (P, m_P)$ be an algebra and $u \in k$. Then $(P, m_P, [-, -]_u)$ is a Poisson algebra, where $[a, b]_u := u(ab - ba)$, for all $a, b \in P$. In particular, any associative algebra $P = (P, m_P)$ is a Poisson algebra with the abelian Lie bracket, i.e. $[a, b] := 0$, for all $a, b \in P$. On the other hand, let $\mathfrak{g} = (\mathfrak{g}, [-, -]_{\mathfrak{g}})$ be a Lie algebra with a linear basis $\{e_i \,|\, i \in I\}$. Then the symmetric algebra $P := S(\mathfrak{g})$ of $\mathfrak{g}$ (i.e. the polynomial algebra $k[e_i \,|\, i \in I]$) is a Poisson algebra with the bracket defined by $[e_i, e_j] := [e_i, e_j]_{\mathfrak{g}}$, for all i, $j \in I$ and extended to the entire algebra $k[e_i \,|\, i \in I]$ via the Leibniz law (43).

If P is a Poisson algebra then the direct product $P \times k$ is a Poisson algebra with the direct product structures of associative/Lie algebra: that is the multiplication and the bracket is defined for any $p, q \in P$ and $x, y \in k$ by:

$$(p, x) \star (q, y) := (pq,\ xy), \qquad \{(p, x),\, (q, y)\} := \big([p, q],\ 0\big) \tag{44}$$

Furthermore, the canonical projection $\pi_P : P \times k \to P$, $\pi_P(p, x) := p$ is a surjective Poisson algebra map having a 1-dimensional kernel. It what follows we shall classify all Poisson algebras Q that admit a surjective Poisson algebra map $Q \to P \to 0$ with a 1-dimensional kernel. In order to do this we first recall from [3] the following concept:

Definition 3.2. A *co-flag datum* of a Poisson algebra P is a 5-tuple $(\lambda, \Lambda, \vartheta, \gamma, f)$, where $\lambda, \Lambda, \gamma : P \to k$ are linear maps, $\vartheta, f : P \times P \to k$ are bilinear maps such that:

(CF1) $(\lambda, \Lambda, \vartheta)$ is a co-flag datum of the first kind of the associative algebra P
(CF2) $\lambda([p, q]) = \Lambda([p, q]) = \gamma([p, q]) = f(p, p) = 0$

(CF3) $f(p,[q,r]) + f(q,[r,p]) + f(r,[p,q]) + \gamma(p)f(q,r) + \gamma(q)f(r,p) + \gamma(r)f(p,q) = 0$
(CF4) $f(pq,\, r) - \Lambda(q)f(p,\, r) - \lambda(p)f(q,\, r) = \gamma(r)\vartheta(p,\, q) + \vartheta([p,\, r],\, q) + \vartheta(p,\, [q,\, r])$
(CF5) $\gamma(pq) = \gamma(p)\Lambda(q) + \lambda(p)\gamma(q)$

for all $p,\, q,\, r \in P$. We denote by $\mathcal{F}(P)$ the set of all co-flag data of P.

The above concept was introduced in [3, Definition 3.2] under the name of 'abelian co-flag datum' of P. We dropped the adjective 'abelian' since the other family of co-flag datum introduced in [3, Definition 3.4] will generate a family of Poisson algebras which are all isomorphic to the direct product $P \times k$ (see the proof of Theorem 3.3 below) – hence, they are irrelevant from the classification view point.

Let $(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f) \in \mathcal{F}(P)$ be a co-flag datum of a Poisson algebra P. Then we shall denote by $P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)} := P \times k$ the direct product of vector spaces which is a Poisson algebra with the multiplication $\star$ and the bracket $\{-,\, -\}$ defined [3, Section 3] for any $p,\, q \in P,\, x,\, y \in k$ by:

$$(p,x) \star (q,y) \;:=\; (pq,\; \vartheta(p,q) + \lambda(p)y + \Lambda(q)x) \tag{45}$$

$$\{(p,x),\, (q,y)\} \;:=\; \big([p,q],\; f(p,q) + \gamma(p)y - \gamma(q)x\big) \tag{46}$$

Now we can prove the following classification result: it is the Poisson version of Theorem 2.8 and improves the classification given in [3, Theorem 3.6] where all Poisson algebras Q having a Poisson surjection $Q \to P \to 0$ with a 1-dimensional kernel are classified in a more restrictive fashion: up to an isomorphism which stabilizes k and co-stabilizes P.

Theorem 3.3. *Let P be a Poisson algebra. Then:*

(1) *A Poisson algebra Q has a surjective Poisson algebra map $Q \to P \to 0$ with a 1-dimensional kernel if and only if $Q \cong P \times k$ or $Q \cong P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}$, for some $(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f) \in \mathcal{F}(P)$.*

(2) *Two Poisson algebras $P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}$ and $P_{(\lambda',\, \Lambda',\, \vartheta',\, \gamma',\, f')}$ are isomorphic if and only if there exists a triple $(s_0,\, \psi,\, r) \in k^* \times \mathrm{Aut}_{\mathrm{Poss}}(P) \times \mathrm{Hom}_k(P,\, k)$ such that for any $p,\, q \in P$:*

$$\lambda = \lambda' \circ \psi, \qquad \Lambda = \Lambda' \circ \psi, \qquad \gamma = \gamma' \circ \psi \tag{47}$$

$$\vartheta(p,q)\, s_0 = \vartheta'\big(\psi(p), \psi(q)\big) + \lambda(p)r(q) + \Lambda(q)r(p) - r(pq) \tag{48}$$

$$f(p,q)\, s_0 = f'\big(\psi(p),\, \psi(q)\big) + \gamma(p)r(q) - \gamma(q)r(p) - r([p,\, q]) \tag{49}$$

(3) *The Poisson algebras $P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}$ and $P \times k$ are not isomorphic.*

Proof. (1) Let Q be a Poisson algebra having a surjective Poisson algebra map $Q \to P \to 0$ with 1-dimensional kernel. Then, using [3, Proposition 2.4 and Proposition 3.5] we obtain that $Q \cong P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}$, for some $(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f) \in \mathcal{F}(P)$ or $Q \cong P^{(\lambda,\, \vartheta,\, u)}$, where $u \in k \setminus \{0\}$, $\lambda : P \to k$ is a unit preserving linear map, $\vartheta : P \times P \to k$ is a bilinear map satisfying the following two compatibilities for any $p,\, q,\, r \in P$:

$$\lambda(pq) = \lambda(p)\lambda(q) - u\,\theta(p,\, q), \quad \theta(p,\, qr) - \theta(pq,\, r) = \theta(p,\, q)\lambda(r) - \theta(q,\, r)\lambda(p) \tag{50}$$

which we called in [3, Definition 3.4] a non-abelian co-flag datum of P. The Poisson algebra $P^{(\lambda,\,\vartheta,\,u)}$ is the vector space $P\times k$ having the multiplication and the Poisson bracket defined by:

$$(p,x)\star(q,y) \;:=\; (pq,\ \vartheta(p,q)+\lambda(p)y+\lambda(q)x+u\,xy) \tag{51}$$

$$\{(p,x),\,(q,y)\} \;:=\; \big([p,q],\ -u^{-1}\,\lambda([p,\,q])\big) \tag{52}$$

for all $p,\,q\in P$ and $x,\,y\in k$. Since $u\neq 0$ we obtain from the first equation of (50) that ϑ is implemented by u and λ by the formula $\vartheta(p,\,q)=u^{-1}\Big(\lambda(p)\lambda(q)-\lambda(pq)\Big)$ and hence the multiplication of the Poisson algebra $P^{(\lambda,\,u)}$ given by (51) takes the form:

$$(p,x)\star(q,y)=(pq,\ u^{-1}\Big(\lambda(p)\lambda(q)-\lambda(pq)\Big)+\lambda(p)y+\lambda(q)x+u\,xy)$$

which is precisely (19). The first part is finished once we observe that the map given by the formula (21), namely $\varphi: P^{(\lambda,\,u)}\to P\times k$, $\varphi(p,x):=(p,\,\lambda(p)+u\,x)$, for all $p\in P$ and $x\in k$ is an isomorphism of Poisson algebras. Indeed, Remark 2.4 shows that φ is an isomorphism of associative algebras; hence we only have to prove that it is also a Lie algebra map, where the bracket on $P^{(\lambda,\,u)}$ is given by (52). A straightforward computation shows that $\varphi\Big(\{(p,\,x),\,(q,\,y)\}\Big)=[\varphi(p,\,x),\,\varphi(q,\,y)]_{P\times k}=\big([p,\,q],\,0\big)$ and thus any Poisson algebra $P^{(\lambda,\,u)}$ is in fact isomorphic to the direct product of Poisson algebras $P\times k$.

(2) The first step in proving Theorem 2.8 gives a bijection between all associative algebra isomorphism corresponding to the Poisson algebras $P_{(\lambda,\,\Lambda,\,\vartheta,\,\gamma,\,f)}$ and $P_{(\lambda',\,\Lambda',\,\vartheta',\,\gamma',\,f')}$ and the set of all triples $(s_0,\,\psi,\,r)\in k^*\times \mathrm{Aut}_{\mathrm{Alg}}(P)\times \mathrm{Hom}_k(P,\,k)$ satisfying (48) and the first two compatibilities of (47). The bijection is given such that the associative algebra isomorphism $\varphi=\varphi_{(s_0,\,\psi,\,r)}: P_{(\lambda,\,\Lambda,\,\vartheta,\,\gamma,\,f)}\to P_{(\lambda',\,\Lambda',\,\vartheta',\,\gamma',\,f')}$ associated to $(s_0,\,\psi,\,r)$ is given by the formula (33), that is $\varphi(p,\,x)=\big(\psi(p),\,r(p)+x\,s_0\big)$, for all $p\in P$ and $x\in k$. The proof will be finished if we show that such a map $\varphi=\varphi_{(s_0,\,\psi,\,r)}$ is also a morphism of Lie algebras if and only if ψ is an automorphism of the Lie algebra $P=(P,\,[-,\,-])$ and the last equation of (47) and (49) hold. This is an elementary fact: by a straightforward computation we can show that $\varphi\Big(\{(p,0),\,(q,0)\}\Big)=\{\varphi(p,0),\,\varphi(q,0)\}$ if and only if $\psi: P\to P$ is a Lie algebra map and (49) holds. In a similar fashion $\varphi\Big(\{(p,0),\,(0,x)\}\Big)=\{\varphi(p,0),\,\varphi(0,x)\}$ if and only if $\gamma=\gamma'\circ\psi$. The rest of the details are left to the reader.

(3) Follows from step (2) of the proof of Theorem 2.8 which proves that the associative algebras $P_{(\lambda,\,\Lambda,\,\vartheta,\,\gamma,\,f)}$ and $P\times k$ are never isomorphic. □

Corollary 3.4. *Let P be a Poisson algebra for which there is no algebra map $P\to k$ or P is perfect as a Lie algebra, i.e. $P=[P,\,P]$. Then, up to an isomorphism, the only Poisson algebra Q for which there exists a surjective algebra map $Q\to P\to 0$ having a 1-dimensional kernel is the direct product $P\times k$ of Poisson algebras.*

Proof. The proof follows from Theorem 3.3 since in both cases the set $\mathcal{F}(P)$ is empty. Indeed, the first case follows form Corollary 2.11, while if P is perfect as a Lie algebra then, using the compatibilities (CF2) we obtain that $\lambda = \Lambda \equiv 0$, the trivial maps, which contradicts the fact that the algebra maps λ and Λ are unit preserving. □

The geometrical meaning of Corollary 3.4 is the following: if M is a real Poisson manifold such that the algebra $C^{\infty}(M)$ of all real smooth functions on M is perfect as a Lie algebra, then up to an isomorphism, there is only one Poisson manifold containing P as a sub-manifold of codimension 1. The group of Poisson algebra automorphisms of $P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}$ can also be described. Using the proof of Theorem 3.3, the Poisson version of Corollary 2.12 takes the following form:

Corollary 3.5. *Let P be a Poisson algebra, $(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f) \in \mathcal{F}\,(P)$ a co-flag datum of P and let $\mathcal{GP}\big(P,\, (\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)\big)$ be the set of all triples $(s_0,\, \psi,\, r) \in k^* \times \mathrm{Aut}_{\mathrm{Poss}}(P) \times P^*$ satisfying the compatibility conditions (47)-(49) written for $\lambda' = \lambda$, $\Lambda' = \Lambda$, $\gamma' = \gamma$, $\vartheta' = \vartheta$ and $f' = f$. Then, there exists an isomorphism of groups*

$$\mathrm{Aut}_{\mathrm{Poss}}\big(P_{(\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)}\big) \cong \mathcal{GP}\,\big(P,\, (\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)\big)$$

where the latter is a group with respect to the following multiplication:

$$(s_0,\, \psi,\, r) \cdot (s_0',\, \psi',\, r') := (s_0 s_0',\, \psi \circ \psi',\, r \circ \psi' + s_0 r')$$

for all $(s_0,\, \psi,\, r)$, $(s_0',\, \psi',\, r') \in\in \mathcal{GP}\,\big(P,\, (\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)\big)$. Moreover, the map

$$\mathcal{GP}\,\big(P,\, (\lambda,\, \Lambda,\, \vartheta,\, \gamma,\, f)\big) \longrightarrow P^* \ltimes_{\zeta} \big(k^* \times \mathrm{Aut}_{\mathrm{Poss}}(P)\big), \qquad (s_0,\, \psi,\, r) \mapsto \big(s_0^{-1} r,\, (s_0,\, \psi)\big)$$

is an injective morphism of groups.

We end the paper with a relevant example which follows by a long computation based on Theorem 3.3; the detailed proof can be provided upon request.

Example 3.6. Let k be a field of characteristic zero and $P := \mathcal{H}(3,\, k)$ the 3-dimensional non-commutative Heisenberg-Poisson algebra: i.e., $\mathcal{H}(3,\, k)$ is the set of all upper triangular 2×2 matrices with the usual multiplication and the Lie bracket given by $[x, y] := xy - yx$. Consider $\{e_{11},\, e_{12},\, e_{22}\}$ a basis of $\mathcal{H}(3,\, k)$ over k. Then the set of isomorphism types of all 4-dimensional Poisson algebras which admit a surjective Poisson algebra map on $\mathcal{H}(3,\, k)$ are the ones listed below together with the usual direct product $\mathcal{H}(3,\, k) \times k$ (we only write down the non-zero products):

$$\begin{aligned}
P_1: \quad & e_{11} \star e_{11} = e_{11},\ e_{11} \star e_{12} = e_{12},\ e_{12} \star e_{22} = e_{12},\ e_{22} \star e_{22} = e_{22},\\
& e_{11} \star f = f \star e_{11} = f,\ \ \{e_{11},\, e_{12}\} = e_{12},\ \{e_{12},\, e_{22}\} = e_{12}.\\
P_2: \quad & e_{11} \star e_{11} = e_{11},\ e_{11} \star e_{12} = e_{12},\ e_{12} \star e_{22} = e_{12},\ e_{22} \star e_{22} = e_{22},\\
& e_{22} \star f = f \star e_{22} = f,\ \ \{e_{11},\, e_{12}\} = e_{12},\ \{e_{12},\, e_{22}\} = e_{12}\\
P_3: \quad & e_{11} \star e_{11} = e_{11},\ e_{11} \star e_{12} = e_{12} - f,\ e_{12} \star e_{11} = f,\ e_{12} \star e_{22} = e_{12} - f\\
& e_{22} \star e_{12} = f,\ e_{22} \star e_{22} = e_{22},\ e_{11} \star f = f \star e_{22} = f,\ \{e_{11},\, e_{12}\} = e_{12},\\
& \{e_{12},\ e_{22}\} = e_{12},\ \{e_{11},\, f\} = f,\ \{e_{22},\, f\} = -f.
\end{aligned}$$

$$
\begin{array}{ll}
P_4^{\omega}: & e_{11}\star e_{11}=e_{11},\ e_{11}\star e_{12}=e_{12},\ e_{12}\star e_{22}=e_{12},\ e_{22}\star e_{22}=e_{22},\\
& e_{11}\star f=f\star e_{22}=f,\ \{e_{11},\, e_{12}\}=e_{12},\ \{e_{12},\ e_{22}\}=e_{12},\\
& \{e_{11},\, f\}=\omega f,\ \{e_{22},\, f\}=-\omega f,\ \text{where } \omega\in k.\\
P_5^{\tau}: & e_{11}\star e_{11}=e_{11},\ e_{11}\star e_{12}=e_{12},\ e_{12}\star e_{22}=e_{12},\ e_{22}\star e_{22}=e_{22},\\
& e_{22}\star f=f\star e_{11}=f,\ \{e_{11},\, e_{12}\}=e_{12},\ \{e_{12},\, e_{22}\}=e_{12},\\
& \{e_{11},\, f\}=\tau f,\ \{e_{22},\, f\}=-\tau f,\ \text{where } \tau\in k.
\end{array}
$$

We point out that, even if up to an isomorphism there are only 8 associative algebras of dimension 4 with a surjective algebra map on the algebra $\mathcal{H}(3,\, k)$ (indicated at the end of Example 2.19), the set of isomorphism types of Poisson algebras having the same algebra structure can be infinite due to the 1-parameter families P_4^{ω} and P_5^{τ}.

Acknowledgement

Parts of this work were undertaken while the first named author was visiting the Max Planck Institute for Mathematics in Bonn. Their hospitality and financial support is gratefully acknowledged.

References

[1] Adem, A. and Milgram, R. J. - Cohomology of finite groups, Springer, 2004.

[2] Agore, A.L. and Militaru, G. - Jacobi and Poisson algebras, *J. Noncommutative Geometry* **9** (2015), 1295–1342.

[3] Agore, A.L. and Militaru, G. - The global extension problem, crossed products and co-flag non-commutative Poisson algebras, *J. Algebra*, **426** (2015), 1–31.

[4] Agore, A.L. and Militaru, G. - Extending structures for Lie algebras, *Monatsh. für Mathematik*, **174** (2014), 169–193.

[5] Agore, A.L. and Militaru, G. - Extending structures, Galois groups and supersolvable associative algebras, *Monatsh. für Mathematik*, DOI:10.1007/s00605-015-0814-8.

[6] Alekseevsky, D., Michor, P. W. and Ruppert, W. - Extensions of super Lie algebras, *J. Lie Theory* **15** (2005), 125–134.

[7] Alperin, J.L. and Bell, R.R. - Groups and representations, Springer-Verlag, New York, 1995.

[8] Andruskiewitsch, N. and Devoto, J. - Extensions of Hopf algebras, *Algebra i Analiz* **7** (1995), 22–61.

[9] Barnes, D.W. and Newell, M.L. - Some theorems on saturated homomorphs of soluble Lie algebras, *Math. Z.*, **115** (1970) 179-187.

[10] Bavula, V.V. and Jordan, D.A. - Isomorphism problems and groups of automorphisms for generalized Weyl algebras, *Trans. Amer. Math. Soc.*, **353** (2001), 769-794.

[11] Brzeziński, T. and Wisbauer, R. - Corings and comodules, London Math Soc. Lect. Note Ser., **309**, Cambridge University Press, 2003

[12] Castiglioni, J. L., Garca-Martnez, X. and Ladra, M. - Universal central extensions of Lie-Rinehart algebras, arXiv:1403.7159

[13] Ceken, S., Palmieri, J., Wang Y.- H. and Zhang, J.J. - The discriminant controls automorphism groups of noncommutative algebras, *Adv. Math.*, **269** (2015), 551–584.

[14] Chevalley, C. and Eilenberg S. - Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.*, **63**(1948), 85–124.

[15] Eilenberg, S. and Maclane, S. - Cohomology theory in abstract groups, II, *Annals of Mathematics*, **48**(1947), 326–341.

[16] Erdmann, K. and Wildon, M.J. - Introduction to Lie algebras, Springer, 2006.

[17] Everett, C. J., Jr. - An extension theory for rings, *Amer. J. Math.*, **64** (1942), 363-370
[18] Farkas, D. R. and Letzter, G. - Ring theory from symplectic geometry, *J. Pure Appl. Algebra*, **125**(1998), 155-190.
[19] Frégier, Y. - Non-abelian cohomology of extensions of Lie algebras as Deligne groupoid, *J. Algebra*, **398** (2014), 243-257.
[20] de Graaf, W.A. - Classification of Solvable Lie Algebras, *Experiment. Math.*, **14** (2005), 15–25.
[21] Grabowski, J., Marmo, G. and Perelomov, A. M. - Poisson structures: towards a classification, *Modern Phys. Lett. A* **8**(1993), 1719–1733.
[22] Grabowski, J. - Brackets, *Int. J. Geom. Methods Mod. Phys.*, 10(8):1360001, 45, 2013.
[23] Janusz, G. J. - Automorphism group of simple algebras and group algebras, Representation Theory of Algebras, Lecture Notes in Pure and Applied Mathematics, Vol. **37** (1976), 381-388.
[24] Huebschmann, J. - Poisson cohomology and quantization, *J. Reine Angew. Math.*, **408** (1990), 57-113.
[25] Huebschmann, J., Perlmutter, M. and Ratiu, T.S. - Extensions of Lie-Rinehart algebras and cotangent bundle reduction, . *Proc. London Math. Soc.*, **107** (2013), 1135–1172.
[26] Hochschild, G. - Cohomology and representations of associative algebras, *Duke Math. J.*, **14**(1947), 921–948.
[27] Horn, R. and Sergeichuk, V. - Canonical matrices of bilinear and sesquilinear forms, *Linear Algebra and its Applications*, **428** (2008), 193-223.
[28] O. Hölder, Bildung zusammengesetzter Gruppen, *Math. Ann.* **46**(1895), 321–422.
[29] Kubo, F. - Finite-dimensional non-commutative Poisson algebras, *J. Pure Applied Algebra*, **113** (1996), 307–314.
[30] Laurent-Gengoux, C., Pichereau, A. and Vanhaecke, P. - Poisson Structures, Vol. 347, 2013, Springer.
[31] Loday, J.-L. and Pirashvili, T. - Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.*, **296**(1993), 139–158.
[32] Milies, C. P. and Sehgal S.K. - An introduction to group rings, Kluwer Academic Publishers, 2002.
[33] Militaru, G. - The global extension problem, co-flag and metabelian Leibniz algebras, *Linear Multilinear Algebra* **63** (2015), 601–621.
[34] Olivieri, A., del Rio, A. and Simón, J. J. - The Group of Automorphisms of the Rational Group Algebra of a Finite Metacyclic Group, *Comm. in Algebra*, **34** (2006), 3543-3567.
[35] Redondo, M. J. - Hochschild cohomology: some methods for computations, IX Algebra Meeting USP/UNICAMP/UNESP (Sao Pedro, 2001), *Resenhas IME-USP*, **5**(2001), 113–137.
[36] Rotman, J.J. - Advanced Modern Algebra, New York: Prentice Hall, 2nd Edition, 2003.
[37] Schreier, O. - Uber die Erweiterung von Gruppen, I, *Monatshefte für Mathematik und Physik*, **34** (1926), 165–180.
[38] Weibel, C.A. - An introduction to homological algebra, Cambridge University Press, 1994.
[39] Williamson, J. - On the algebraic problem concerning the normal forms of linear dynamical systems, *Amer. J. Math.*, **58** (1936), 141–163.

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium **and** "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

E-mail address: ana.agore@vub.ac.be and ana.agore@gmail.com

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com